1. Show that the series
\[ w(z) := 1 + \left( \frac{1}{2} \right)^2 z + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 z^2 + \cdots + \left( \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \right)^2 z^n + \cdots \]

satisfies the differential equation
\[ z(1-z)w'' + (1-2z)w' - w/4 = 0. \]

The Chebyshev polynomials are defined as follows: put \( x = \cos(\phi) \), then \( T_n(x) := \cos(n\phi) \), \( U_n(x) := \frac{T_{n+1}(x)}{n+1} \).

2. Show that all roots of \( T_n \) and \( U_n \) are in \([-1, 1]\) and determine these roots.

3. Show that \( T_n \) and \( U_n \) satisfy the differential equations
\[
(1 - x^2)T''_n(x) - xT'_n(x) + n^2 T_n(x) = 0 \]
\[
(1 - x^2)U''_n(x) - 3xU'_n(x) + n(n+2)U_n(x) = 0 \]

4. Let \( P_n \) be the \( n \)-th Legendre polynomial. Show that
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

5. Show that
\[
P_n(x) = \frac{1}{\pi} \int_{0}^{\pi} \left( x + \sqrt{x^2 - 1} \cos(\phi) \right)^n d\phi.
\]