1. Introduction

In this paper we study an asymptotic filling invariant \( \widehat{\text{div}}_k \) similar to the \( k \)-th divergence \( \text{div}_k \) defined by Brady and Farb \cite{BrFa} and show that it can be used to detect the Euclidean rank of all proper cocompact Hadamard spaces in the sense of Alexandrov, i.e. proper cocompact CAT(0)-spaces. We thereby extend results of \cite{BrFa, Leu, Hin} from the setting of symmetric spaces of non-compact type to that of singular Hadamard spaces. We furthermore exhibit the optimal power for the growth for \( \widehat{\text{div}}_k \) for symmetric spaces of non-compact type with Euclidean rank no larger than \( k \) and for CAT(\( \kappa \))-spaces with \( \kappa < 0 \).

In \cite{BrFa} Brady and Farb introduced a new quasi-isometry invariant \( \text{div}_k(X) \) of a cocompact Hadamard manifold \( X \). The \( k \)-dimensional divergence \( \text{div}_k(X) \) can be seen as a higher dimensional analogue of the divergence of geodesics (studied by Gersten in \cite{Ger1} and \cite{Ger2}) and in some sense measures the \( (k + 1) \)-dimensional spread of geodesic rays in \( X \). Brady and Farb then proved that \( \text{div}_{k-1}(X) \) has exponential growth when \( X = H^{m_1} \times \cdots \times H^{m_k} \) is the product of \( k \) hyperbolic planes. The main idea in their proof was to construct a family of quasi-isometric embeddings of \( H^{m_1+\cdots+m_k-k+1} \) in \( X \) transversal to a maximal flat. Using the same idea Leuzinger \cite{Leu} extended this result to symmetric spaces of non-compact type. In particular, he showed that \( \text{div}_k(X) \) grows exponentially for \( k = \text{Rank}X - 1 \), where \( \text{Rank}X \) denotes the Euclidean rank, i.e. the maximal dimension of an isometrically embedded Euclidean space in \( X \). In \cite{BrFa} the authors asked whether \( \text{div}_k(X) \) can be used to detect the Euclidean rank \( \text{Rank}X \) of a symmetric space \( X \). This question has recently been answered in the affirmative by Hindawi in \cite{Hin}. The primary aim of the note here is to show that a slightly modified version of \( \text{div}_k(X) \) can be used to detect the Euclidean rank of all proper cocompact Hadamard spaces \( X \). Our invariant \( \widehat{\text{div}}_k(X) \) is defined using integral currents instead of Lipschitz maps as is done in \cite{BrFa} but is otherwise unchanged. We use the theory of integral currents.
currents in metric spaces developed by Ambrosio and Kirchheim in [AmKi1]. Roughly speaking, $\hat{\text{div}}_k(X)$ is a three-parameter family of functions $\delta^{k}_{x_0, \varrho, A}(r)$ where $x_0 \in X$, $A > 0$, and $0 < \varrho < 1$. For fixed parameters, $\delta^{k}_{x_0, \varrho, A}(r)$ is defined to be the maximal mass of an integral $(k + 1)$-current with support outside the open ball $U(x_0, \varrho r)$ needed to fill a $k$-dimensional integral cycle with support in the metric sphere $S(x_0, r)$ and with mass at most $A r^k$. Precise definitions will be given in Section 2. In order to distinguish different types of growth we use the following convention.

**Definition 1.1.** Let $X$ be a complete metric space, $k \in \mathbb{N}$, and $\beta \in [1, \infty)$. We write $\hat{\text{div}}_k(X) \preceq r^\beta$ if there exist $0 < \varrho_0 \leq 1$ such that
\[
\limsup_{r \to \infty} \frac{\delta^{k}_{x_0, \varrho_0, A}(r)}{r^\beta} < \infty
\]
for all $x_0 \in X$ and all $A > 0$. On the other hand, we write $\hat{\text{div}}_k(X) \succeq r^\beta$ if there exists $A_0 > 0$ such that
\[
\liminf_{r \to \infty} \frac{\delta^{k}_{x_0, \varrho, A_0}(r)}{r^\beta} > 0
\]
for all $x_0 \in X$ and all $\varrho \in (0, 1)$.

It is not difficult to see that if $X$ is a Hadamard space then the growth of $\delta^{k}_{x_0, \varrho, A}$ is in some sense independent of the basepoint $x_0$, see Lemma 2.4.

**Remark 1.2.** The following quasi-isometry invariance property for $\text{div}_k$ was proved in [BrFa] and can easily be established for $\hat{\text{div}}_k$ by the same methods. Let $\varphi : X \to Y$ be a quasi-isometry between cocompact Hadamard manifolds and let $\beta \in [k + 1, \infty)$. If $\hat{\text{div}}_k(X) \preceq r^\beta$ then $\hat{\text{div}}_k(Y) \preceq r^\beta$. On the other hand, if $\hat{\text{div}}_k(X) \succeq r^\beta$ then $\hat{\text{div}}_k(Y) \succeq r^\beta$. In this sense $\hat{\text{div}}_k$ is a quasi-isometry invariant for cocompact Hadamard manifolds, see also Section 2.

Our main result can be stated as follows.

**Theorem 1.3.** Let $X$ be a proper cocompact Hadamard space and let $k \in \mathbb{N}$. If $k = \text{Rank} X - 1$ then $\hat{\text{div}}_k(X) \succeq r^{k+2}$. On the other hand, if $k \geq \text{Rank} X$ then $\hat{\text{div}}_k(X) \preceq r^{k+1}$.

As a consequence we obtain the following generalization of the main result in [Hin].

**Corollary 1.4.** The $\hat{\text{div}}_k$ can be used to detect the Euclidean rank of all proper cocompact Hadamard spaces.

The estimates in Theorem 1.3 are good enough to detect the Euclidean rank of every proper cocompact Hadamard space but the optimal growth rate of $\hat{\text{div}}_k$ is believed to be different.

**Question:** Let $(X, d)$ be a proper cocompact Hadamard space. Is it true that
(i) the divergence $\widehat{\text{div}}_k(X)$ grows exponentially if $k = \text{Rank} X - 1$?
(ii) the divergence $\widehat{\text{div}}_k(X)$ grows polynomial of degree $k$ if $k \geq \text{Rank} X$?

As mentioned above, for symmetric spaces of non-compact type (i) follows from [Leu]. In the following we give an answer to (ii) when $X$ is a symmetric space of non-compact type or a complete CAT($\kappa$)-space with $\kappa < 0$. These results are consequences of a simple relation between $\widehat{\text{div}}_k(X)$ and the type of isoperimetric inequality in $X$. To state the results we adopt the following notation.

**Definition 1.5.** Let $k \in \mathbb{N}$, $\alpha \in [1, \frac{k+1}{k}]$ and let $X$ be a complete metric space. We say that $X$ admits an isoperimetric inequality of power $\alpha$ for $I_k(X)$ if there exists a constant $C$ such that for every $T \in I_k(X)$ with $\partial T = 0$ there is an $S \in I_{k+1}(X)$ with $\partial S = T$ and
$$M(S) \leq C[M(T)]^\alpha.$$ 

In the above definition, $I_k(X)$ denotes the space of $k$-dimensional metric integral currents introduced in [AmKi1]. See Section 2 for the definition.

**Remark 1.6.** Using the arguments of [BrFa] it can be shown, that the power $\alpha$ of the isoperimetric inequality for $I_k(X)$ is a quasi-isometry invariant for cocompact Hadamard manifolds. However, in what follows, we do not stress this aspect.

In [Wen1] it was shown that every Hadamard space $X$ admits an isoperimetric inequality of at least Euclidean type for $I_k(X)$, i.e. with power at most $\alpha := \frac{k+1}{k}$. As for CAT($\kappa$)-spaces with $\kappa < 0$ we prove the following linear isoperimetric inequality.

**Theorem 1.7.** Let $(X, d)$ be a complete CAT($\kappa$)-space with $\kappa < 0$. Then for every $k \geq 1$ and every $T \in I_k(X)$ with $\partial T = 0$ and bounded support there exists $S \in I_{k+1}(X)$ with $\partial S = T$ and such that
$$M(S) \leq \frac{1}{\sqrt{-\kappa k}} M(T).$$

This result is well-known and easy to prove in the case of simply-connected Riemannian manifolds of negative sectional curvature. The proof of Theorem 1.7 relies on the same constructions. However, the estimates are not as straight-forward as in the smooth case.

We now state a simple relation between the isoperimetric inequality and the growth of $\widehat{\text{div}}_k(X)$.

**Proposition 1.8.** Let $X$ be a Hadamard space and $k \in \mathbb{N}$. If $X$ admits an isoperimetric inequality of power $\alpha < \frac{k+1}{k}$ for $I_k(X)$ then $\widehat{\text{div}}_k(X) \preceq r^{\alpha k}$.

Using the above theorem we exhibit the optimal growth rate for $\widehat{\text{div}}_k(X)$ for symmetric spaces $X$ of non-compact type and for complete CAT($\kappa$)-spaces $X$ with $\kappa < 0$. This generalizes results in [Hin].
Corollary 1.9. If $X$ is a complete CAT($\kappa$)-space with $\kappa < 0$ then $\hat{\text{div}}_k(X) \preceq r^k$ for all $k \in \mathbb{N}$.

This is an immediate consequence of Proposition 1.8 and Theorem 1.7.

Corollary 1.10. If $X$ is a symmetric space of non-compact type then $\hat{\text{div}}_k(X) \preceq r^k$ for all $k \geq \text{Rank}(X)$.

This follows from Proposition 1.8 and the well-known fact that symmetric spaces $X$ of non-compact type admit a linear isoperimetric inequality for $I_k(X)$ for $k \geq \text{Rank}(X)$. The latter is a consequence of the fact that the orthogonal projection onto maximal flats in $X$ decreases the $k$-dimensional volume exponentially with the distance to the flat.

The structure of the paper is as follows: In Section 2 we recall the necessary definitions concerning Hadamard spaces and integral currents and then define the divergence $\hat{\text{div}}_k$. Section 3 contains the proof of Theorem 1.3. The purpose of Section 4 is to prove Theorem 1.7 and Proposition 1.8.

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2. Preliminaries

The purpose of this section is to fix notation regarding CAT($\kappa$)-spaces and metric integral currents on the one hand and to define the new divergence invariants $\hat{\text{div}}_k(X)$ on the other hand.

2.1. Metric spaces with upper curvature bounds. For a general reference on metric spaces of curvature bounded above in the sense of Alexandrov we refer the reader to [Bal], [BrHa], and [BBI]. The notation we use in this article is consistent with [BrHa]. Let $\kappa \in \mathbb{R}$ and set $D_\kappa := \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$ and $D_\kappa := \infty$ otherwise. We note that $D_\kappa = \text{diam}(\mathcal{M}_2^2)$ where $\mathcal{M}_2^2$ is the 2-dimensional simply-connected Riemannian manifold of constant sectional curvature $\kappa$. A metric space $(X, d)$ is called CAT($\kappa$) if the following two properties hold:

(i) $X$ is $D_\kappa$-geodesic: Any two points $x, y \in X$ with $d(x, y) < D_\kappa$ can be joined by a geodesic, i.e. a curve of length $d(x, y)$.

(ii) Every geodesic triangle in $X$ of perimeter $< 2D_\kappa$ satisfies the CAT($\kappa$)-inequality, i.e. it is at least as slim as a comparison triangle in $\mathcal{M}_2^2$.

We refer the reader to [BrHa, Definition II.1.1] for the precise definition of (ii). Following [Bal] we call complete CAT(0)-spaces Hadamard spaces. Furthermore, by definition, a metric space $X$ is called an Alexandrov space of curvature bounded from above by $\kappa$ is if for every point $x \in X$ there is a closed ball $B(x, r)$ which is CAT($\kappa$). In the following we will write $B(x, r)$ for the closed ball $\{ x' \in X : d(x, x') \leq r \}$ and $U(x, r)$ for the open ball
\{x' \in X : d(x, x') < r\}. Furthermore, \(S(x, r)\) will denote the metric sphere \(\{x' \in X : d(x, x') = r\}\).

Recall that \(X\) is said to be cocompact if there is a group \(\Gamma\) of isometries of \(X\) and a compact set \(K \subset X\) such that \(X = \bigcup_{g \in \Gamma} gK\).

**Definition 2.1.** The Euclidean rank of a cocompact Hadamard space \(X\) is defined to be

\[
\text{Rank}_X := \sup \{n \in \mathbb{N} : \text{There exists an isometric embedding } \mathbb{R}^n \hookrightarrow X\}.
\]

In Section 4 we will need the following characterization of a CAT(\(\kappa\))-space, see [BrHa, Proposition II.1.11]. A \(D\kappa\)-geodesic metric space \(X\) is CAT(\(\kappa\)) if and only if every 4-tuple of points \(x_1, x_2, x_3, x_4 \in X\) with

\[
d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_1) < 2D\kappa
\]

has a subembedding in \(M^{2\kappa}\) in the following sense: There exist points \(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4 \in M^{2\kappa}\) such that

\[
\overline{d}(\overline{x}_i, \overline{x}_{i+1}) = d(x_i, x_{i+1}) \quad \text{for } i \mod 4
\]
as well as \(d(x_1, x_3) \leq \overline{d}(\overline{x}_1, \overline{x}_3)\) and \(d(x_2, x_4) \leq \overline{d}(\overline{x}_2, \overline{x}_4)\).

**2.2. Lipschitz maps into metric spaces.** Let \((X, d)\) be a metric space, \(U \subset \mathbb{R}^k\) open, and let \(\varphi : U \to X\) be a Lipschitz map. In [Kir] Kirchheim proved that for almost every \(z \in U\) the metric differential

\[
\text{md } \varphi_z(v) := \lim_{r \searrow 0} \frac{d(\varphi(z + rv), \varphi(z))}{r}
\]
exists for every \(v \in \mathbb{R}^k\) and is a semi-norm on \(\mathbb{R}^k\). This was independently discovered by Korevaar and Schoen, see [KoSc]. It can be shown that for almost every \(z \in U\)

\[
\lim_{r \searrow 0} \frac{1}{r} d(\varphi(z + rv), \varphi(z + rw)) = \text{md } \varphi_z(v - w)
\]

for all \(v, w \in \mathbb{R}^k\). If \(U \subset \mathbb{R}^k\) is merely measurable then \(\text{md } \varphi_z\) can be defined at almost every Lebesgue density point \(z \in U\) by a simple approximation argument. We denote by \(\mathcal{H}^k_0\) or \(\mathcal{H}^k_X\) the \(k\)-dimensional Hausdorff measure on \(X\) given by

\[
\mathcal{H}^k(A) := \lim_{\delta \searrow 0} \inf \left\{ \sum_{i=1}^{\infty} \omega_k \left( \frac{\text{diam}(B_i)}{2} \right)^k : A \subset \bigcup_{i=1}^{\infty} B_i, \text{diam}(B_i) < \delta \right\}
\]
for \(A \subset X\) where \(\omega_k\) is the Lebesgue measure of the unit ball in \(\mathbb{R}^k\). If there is no danger of ambiguity we will write \(\mathcal{H}^k(A)\) for \(\mathcal{H}^k_0(A)\). The \(k\)-th Jacobian of a semi-norm \(s\) on \(\mathbb{R}^k\) is defined by

\[
J_k(s) := \frac{\omega_k}{\mathcal{H}^k(\{ v \in \mathbb{R}^k : s(v) \leq 1 \})}.
\]
If \( s \) is a norm then one easily checks that
\[
J_k(s) = \frac{\mathcal{H}^k_s(Q)}{\mathcal{H}^k_{eucl}(Q)}
\]
whenever \( Q \subset \mathbb{R}^k \) has strictly positive measure. The area factor of \( s \) is given by
\[
\lambda_s := \max \left\{ \frac{\mathcal{H}^k(L([0,1]^k))}{J_k(s)} : L = (L_1, \ldots, L_k) : (\mathbb{R}^k, s) \to \mathbb{R}^k \text{ lin}, L_i \text{ 1-lip} \right\}.
\]
It can be shown that \( k^{-k/2} \leq \lambda_s \leq 2^k/\omega_k \) for every norm \( s \) on \( \mathbb{R}^k \), see [AmKi1, Lemma 9.2]. If \( s \) comes from an inner product then clearly \( \lambda_s = 1 \).

It should be noted that for a normed space \((V, \| \cdot \|)\) the \( k \)-volume density
\[
\mu(v_1 \wedge \cdots \wedge v_k) := \lambda_{\| \cdot \| \text{span}(v_1, \ldots, v_k)} \mathcal{H}_V^k(v_1 \wedge \cdots \wedge v_k)
\]
is usually called Gromov mass* or Benson volume density, see e.g. [AlTh].

2.3. Integral currents in metric spaces. The general reference for the this section is the work of Ambrosio and Kirchheim [AmKi1] where the theory of normal and integral currents is extended from the setting of Euclidean space to arbitrary metric spaces. The classical theory was developed to a large part by Federer and Fleming, see [FeFl] and [Fed]. Let \((X, d)\) be a complete metric space, \( k \geq 0 \), and let \( D^k(X) \) denote the set of \((k+1)\)-tuples \((f, \pi_1, \ldots, \pi_k)\) of Lipschitz functions on \( X \) with \( f \) bounded. The Lipschitz constant of a Lipschitz function \( f \) will be denoted by \( \text{Lip}(f) \). The space of Lipschitz functions on \( X \) will be denoted by \( \text{Lip}(X) \). A \( k \)-dimensional metric current \( T \) on \( X \) is a multi-linear functional on \( D^k(X) \) which is continuous, satisfies a locality property and is of finite mass in the following sense. There exists a finite Borel measure \( \mu \) on \( X \) such that
\[
|T(f, \pi_1, \ldots, \pi_k)| \leq \prod_{i=1}^k \text{Lip}(\pi_i) \int_X |f|d\mu
\]
for all \((f, \pi_1, \ldots, \pi_k) \in D^k(X)\). For the precise definition of the continuity and the locality property we refer the reader to Definition 3.1 in [AmKi1]. Formally, the tuples \((f, \pi_1, \ldots, \pi_k) \in D^k(X)\) should replace a differential form of the kind \( fd\pi_1 \wedge \cdots \wedge d\pi_k \) whenever such an expression makes sense. Continuity of \( T \) is defined with respect to pointwise convergence with bounded Lipschitz constants. The locality property in some sense forces \( T \) to depend on the ‘derivatives’ of the \( \pi_i \) rather than on the \( \pi_i \) themselves. The mass of \( T \) is by definition the smallest Borel measure \( \mu \) satisfying (2) and is denoted by \( \|T\| \). The number \( M(T) := \|T\|(X) \) is also called mass of \( T \). The support of \( T \) is the closed set
\[
\text{spt } T := \{x \in X : \|T\|(B(x, r)) > 0 \text{ for all } r > 0\}.
\]
As is done in [AmKi1] we assume throughout this paper that the cardinality of any set is an Ulam number. This is consistent with the standard ZFC set
We then have that \( \text{spt} T \) is separable and that \( \|T\| \) is concentrated on a \( \sigma \)-compact set. Since the space of bounded Lipschitz functions is dense in \( L^\infty(X,\|T\|) \) it follows from (2) that \( T \) can be extended to a functional on \( L^\infty(X,\|T\|) \times \text{Lip}^k(X) \). An example of a \( k \)-dimensional metric current on \( \mathbb{R}^k \) is given by

\[
[\theta](f, \pi_1, \ldots, \pi_k) := \int_K \theta f \det \left( \frac{\partial \pi_i}{\partial x_j} \right) d\mathcal{L}^k
\]

for all \( (f, \pi_1, \ldots, \pi_k) \in D^k(\mathbb{R}^k) \), where \( K \subset \mathbb{R}^k \) is measurable and \( \theta \in L^1(K,\mathbb{R}) \). If \( \varphi : X \to Y \) is a Lipschitz map between metric spaces then the push-forward under \( \varphi \) of a \( k \)-dimensional current \( T \) on \( X \) is given by

\[
\varphi_# T(g, \tau_1, \ldots, \tau_k) := T(g \circ \varphi, \tau_1 \circ \varphi, \ldots, \tau_k \circ \varphi)
\]

for all \( (g, \tau_1, \ldots, \tau_k) \in D^k(Y) \). This defines a \( k \)-dimensional metric current on \( Y \). The restriction of \( T \) to a Borel set \( A \subset X \) is defined by

\[
(T \llcorner A)(f, \pi_1, \ldots, \pi_k) := T(f \chi_A, \pi_1, \ldots, \pi_k).
\]

This is well-defined. The boundary of \( T \) is the functional on \( D^{k-1}(X) \) defined by

\[
\partial T(f, \pi_1, \ldots, \pi_{k-1}) := T(1, f, \pi_1, \ldots, \pi_{k-1})
\]

for \( (f, \pi_1, \ldots, \pi_{k-1}) \in D^{k-1}(X) \). In general, \( \partial T \) satisfies the continuity and locality property but may fail to satisfy the finite mass condition. If \( \partial T \) also has finite mass (i.e. is a \((k-1)\)-dimension current) then \( T \) is called a normal current and the corresponding space is denoted by \( \mathcal{N}^k(X) \).

In this article we will mainly work with integral currents. We first recall the definition of integer rectifiable currents. We recall that an \( \mathcal{H}^k \)-measurable set \( A \subset X \) is said to be countably \( \mathcal{H}^k \)-rectifiable if there exist countably many Lipschitz maps \( \varphi_i : B_i \to X \) where \( B_i \subset \mathbb{R}^k \) such that

\[
\mathcal{H}^k \left( A \setminus \bigcup_{i=1}^\infty \varphi_i(B_i) \right) = 0.
\]

A 0-dimensional metric current \( T \) is called integer rectifiable if there are exist finitely many points \( x_1, \ldots, x_n \in X \) and \( \theta_1, \ldots, \theta_n \in \mathbb{Z} \) such that

\[
T(f) = \sum_{i=1}^n \theta_i f(x_i)
\]

for all bounded Lipschitz functions \( f \).

**Definition 2.2.** A \( k \)-dimensional metric current \( T \) with \( k \geq 1 \) is said to be integer rectifiable if the following properties hold:

(i) \( \|T\| \) is concentrated on a countably \( \mathcal{H}^k \)-rectifiable set and vanishes on \( \mathcal{H}^k \)-negligible Borel sets.

(ii) For any Lipschitz map \( \varphi : X \to \mathbb{R}^k \) and any open set \( U \subset X \) there exists \( \theta \in L^1(\mathbb{R}^k,\mathbb{Z}) \) such that \( \varphi_#(T \llcorner U) = [\theta] \).
A $k$-dimensional normal current which is in addition integer rectifiable is called an integral current. The corresponding space is denoted by $I_k(X)$. The boundary of an integral current is again an integral current as follows from Theorem 8.6 (boundary rectifiability theorem) of [AmKi1].

We end this section with the following product construction defined in [Wen1]. It is a straightforward generalization of the cone construction given in [AmKi1]. For this endow $[0, 1] \times X$ with the Euclidean product metric and let $f \in \text{Lip}([0, 1] \times X)$. For $x \in X$ and $t \in [0, 1]$ we write $f_t(x) := f(t, x)$. To $T \in N_k(X)$ and $t \in [0, 1]$ we associate a $k$-dimensional normal current on $[0, 1] \times X$ by

\[
([t] \times T)(f, \pi_1, \ldots, \pi_k) := T(f_t, \pi_{1t}, \ldots, \pi_{kt})
\]

for $(f, \pi_1, \ldots, \pi_k) \in D^k([0, 1] \times X)$. We also associate to $T$ the functional

\[
([0, 1] \times T)(f, \pi_1, \ldots, \pi_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 T \left( f_t \frac{\partial \pi_{it}}{\partial t} + \pi_{1t}, \ldots, \pi_{i-1t}, \pi_{i+1t}, \ldots, \pi_{kt+1} \right) \ dt
\]

for $(f, \pi_1, \ldots, \pi_{k+1}) \in D^{k+1}([0, 1] \times X)$. It can be checked (see [Wen1] and [AmKi1]) that $[0, 1] \times T \in N_{k+1}([0, 1] \times X)$ and

\[
(3) \quad \partial([0, 1] \times T) = [1] \times T - [0] \times T - [0, 1] \times \partial T.
\]

Furthermore, if $T \in I_k(X)$ then $[0, 1] \times T \in I_{k+1}(X)$.

### 2.4. The $k$-th divergence $\widehat{\text{div}}_k(X)$.

**Definition 2.3.** Let $X$ be a complete metric space and $k \geq 0$. Then the $k$-th divergence $\widehat{\text{div}}_k(X)$ of $X$ is defined to be the three parameter family of functions

\[
\widehat{\text{div}}_k(X) := \left\{ \delta^k_{x_0, \theta, A} : x_0 \in X, \, 0 < \theta \leq 1, \, A > 0 \right\}
\]

where $\delta^k_{x_0, \theta, A}(r)$ is given by

\[
\delta^k_{x_0, \theta, A}(r) = \sup \left\{ \text{Fillvol}_{X \setminus U(x_0, \theta r)}(T) : T \in I_k(X), \, \text{spt} \ T \subset S(x_0, r) \text{ cpt}, \right.
\]

\[
\partial T = 0, \quad \mathbf{M}(T) \leq Ar^k \bigg\}.
\]

Here, for a closed subset $A \subset X$ and a $T \in I_k(X)$ with spt $T \subset A$ the filling volume of $T$ in $A$ is defined as

\[
\text{Fillvol}_A(T) := \inf \left\{ \mathbf{M}(S) : S \in I_{k+1}(X), \, \partial S = T, \, \text{spt} \ S \subset A \right\}
\]

where we agree on inf $0 = \infty$.

In this paper we will mainly work with the growth types defined in Definition 1.1. For completeness we recall the general definitions from the work of Brady and Farb, in particular the invariance under quasi-isometries. Given $f, g : [0, \infty) \to [0, \infty)$ and $\beta \in [1, \infty)$ we write $f \preceq_{\beta} g$ if there exist constants
Lemma 2.4. Let $(X, d)$ be a Hadamard space and $A > 0$, $\varrho \in (0, 1)$. Then

\[
\delta_{x_{0'}, A} \preceq \delta_{x_{0'}, A'} k \text{ for all } x_{0'}, x'_{0} \in X, \text{ all } g' > \varrho \text{ and } A' > A.
\]

Proof. Fix $\varrho, \varrho', A, A'$ as in the hypothesis. For $x_{0}, x'_{0} \in X$ set

\[ L := d(x_{0}, x'_{0}). \]

Let $T \in I_k(X)$ with $\partial T = 0$ and such that $\text{spt } T \subset S(x_{0}, r)$ and $M(T) \leq A r^k$ where $r$ is large enough. Set $r' := r - L$ and denote by $\pi : X \to B(x'_{0}, r')$ the nearest point projection. Define $\varphi : [0, 1] \times X \to B(x'_{0}, r')$ to be the locally Lipschitz map for which $t \mapsto \varphi(t, x)$ is the constant speed geodesic from $\pi(x)$ to $x$. Set $S' := \varphi_{\#}([0, 1] \times T)$ and note that $S' \in I_{k+1}(X)$ and, by (3), we furthermore have $\partial S' = T - \pi_{\#} T$. For $(f, \tau_1, \ldots, \tau_{k+1}) \in D^{k+1}(X)$ we abbreviate

\[
(f^\delta, \hat{\tau}_1, \ldots, \hat{\tau}_k) := (f \circ \varphi_1 \circ \varphi_2 \circ \ldots \circ \varphi_{k+1}),
\]

and compute

\[
\left| S'(f, \tau_1, \ldots, \tau_{k+1}) \right| \leq \sum_{i=1}^{k+1} \left| \int_0^1 T \left( f \frac{\partial \hat{\tau}_i}{\partial t}, \hat{\tau}_1, \ldots, \hat{\tau}_{i-1}, \hat{\tau}_{i+1}, \ldots, \hat{\tau}_{k+1} \right) dt \right|.
\]

\[
\leq \sum_{i=1}^{k+1} \int_0^1 \prod_{j \neq i} \text{Lip}(\hat{\tau}_j) \int_X \left| f \frac{\partial \hat{\tau}_i}{\partial t} \right| d\|T\| dt.
\]

\[
\leq 2(k+1)L \prod_{j=1}^{k+1} \text{Lip}(\tau_j) \int_0^1 \int_X |f \circ \varphi(x, t)| d\|T\| dt.
\]

From this it follows that $\|S'\| \leq 2(k+1)L \varphi_{\#} (L^1 \times \|T\|)$ and in particular

\[
M(S') \leq 2(k+1)L M(T) \leq 2(k+1)L A r^k.
\]

If $r$ is chosen large enough we furthermore have

\[
M(\pi_{\#} T) \leq M(T) \leq A r^k \leq A' r'^k.
\]

Since also $\text{spt } \pi_{\#} T \subset S'(x_{0}, r')$ there exists by definition $S'' \in I_{k+1}(X)$ satisfying $\partial S'' = \pi_{\#} T$ as well as $M(S'') \leq \delta_{x_{0'}, A'} k$ and $\text{spt } S'' \subset X \setminus U(x'_{0}, \varrho r')$. It follows that

\[
S := S' + S'' \in I_{k+1}(X), \quad \partial S = T \quad \text{and} \quad M(S) \leq \delta_{x_{0'}, A'} k + 2L(k+1)A r^k.
\]

If $r$ is chosen large enough then one easily verifies that furthermore $\text{spt } S \subset X \setminus U(x_{0}, \varrho r)$. This completes the proof. \qed

Definition 2.5. Given two complete metric spaces $X$ and $X'$ and $\beta \in [1, \infty)$ we say that $\hat{\text{div}}(X) \preceq_\beta \hat{\text{div}}(X')$ if there exist $0 < \varrho_0, \varrho'_0 \leq 1$ and $A_0, A'_0 > 0$ such that for every triple $(x_{0}, \varrho, A)$ with $x_{0} \in X$, $\varrho \leq \varrho_0$ and $A \geq A_0$ there exist $x'_{0} \in X'$, $\varrho' \leq \varrho'_0$ and $A' \geq A'_0$ with $\delta_{x_{0}, \varrho, A} \preceq_\beta \delta_{x'_{0}, \varrho', A'}$ where the two functions represent the growth functions for $X$ and $X'$, respectively.
If $\hat{\text{div}}_k(X) \preceq_\beta \hat{\text{div}}_k(X')$ and $\hat{\text{div}}_k(X') \preceq_\beta \hat{\text{div}}_k(X)$ then we write $\hat{\text{div}}_k(X) \sim_\beta \hat{\text{div}}_k(X')$.

In [BrFa] it was shown that $\text{div}_k$ is a quasi-isometry invariant in the class of cocompact Hadamard manifolds. As mentioned above, one can analogously prove that $\hat{\text{div}}_k$ is a quasi-isometry invariant for cocompact Hadamard manifolds. This fact will not be used in the proofs of the other results.

**Proposition 2.6.** If $X$ and $Y$ are $n$-dimensional quasi-isometric Hadamard manifolds then $\hat{\text{div}}_k(X) \sim_{k+1} \hat{\text{div}}_k(Y)$ for all $1 \leq k \leq n-2$.

**Proof.** This follows exactly as for $\text{div}_k(X)$ in [BrFa].  

It seems to be unknown whether $\hat{\text{div}}_k(X) \sim_{k} \hat{\text{div}}_k(Y)$ in the proposition above. This is somewhat unsatisfactory.

3. PROOF OF THE MAIN RESULT

The aim of this section is to prove Theorem 1.3. For its proof we will need the following lemma which is a variation of an argument of Ambrosio and Kirchheim [AmKi1], see also [Wen1, Lemma 3.4]. It shows the existence of ‘fillings’ with high volume growth.

**Lemma 3.1.** Let $X$ be a Hadamard space, $k \in \mathbb{N}$, and $\alpha \in [1, \frac{k+1}{k}]$. Suppose that $X$ admits an isoperimetric inequality of power $\alpha$ for $\text{I}_k(X)$ and set $C := \max\{C', C''\}$ where $C'$ and $C''$ are the constants of the isoperimetric inequalities of power $\alpha$ and $\frac{k+1}{k}$, respectively. Then there exists for every $T \in \text{I}_k(X)$ with $\partial T = 0$ an $S \in \text{I}_{k+1}(X)$ with $\partial S = T$ and

$$M(S) \leq C[M(T)]^\alpha$$

and which has the following growth: Whenever $x \in \text{spt} S$ and $0 \leq s \leq \text{dist}(x, \text{spt} T)$ then

$$\|S\|(B(x, s)) \geq \frac{s^{k+1}}{(3C)^k(k+1)^{k+1}}.$$  

Moreover, if $3C(k+1) \leq s \leq \text{dist}(x, \text{spt} T)$ then

$$\|S\|(B(x, s)) \geq \begin{cases} 3C \left\{ 1 + \frac{\alpha-1}{\alpha} \left[ \frac{s}{3C} - (k+1) \right] \right\} \frac{s^{\alpha-1}}{C \exp(\frac{s}{3C} - (k+1))} & \text{if } \alpha > 1 \\ 3C \exp(\frac{s}{3C} - (k+1)) & \text{if } \alpha = 1. \end{cases}$$

It will be clear from the proof that all the conclusions hold also true for absolute area-minimizers $S \in \text{I}_{k+1}(X)$ with $\partial S = T$.

**Proof.** Let $\mathcal{M}$ denote the complete metric space consisting of all $S \in \text{I}_{k+1}(X)$ with $\partial S = T$ and endowed with the metric given by $d_M(S, S') := M(S - S')$. Choose an $\tilde{S} \in \mathcal{M}$ satisfying $M(\tilde{S}) \leq C[M(T)]^\alpha$. By Ekeland’s variation principle [Eke] there exists an $S \in \mathcal{M}$ with $M(S) \leq M(\tilde{S})$ and such that the function

$$S' \mapsto M(S') + \frac{1}{2} M(S - S')$$
has an absolute minimum at \( S' = S \). Let \( x \in \text{spt} S' \setminus \text{spt} T \) and set \( q_x(y) := d(x, y) \). Then the slicing theorem implies that for almost every \( 0 < s < \text{dist}(x, \text{spt} T) \) the slice \( \langle S, q_x, s \rangle \) exists, has zero boundary, and belongs to \( \mathbf{I}_k(X) \). For an \( S_\alpha \in \mathbf{I}_{k+1}(X) \) with \( \partial S_\alpha = \langle S, q_x, s \rangle \) the integral current \( S \uplus (X \setminus B(x, s)) + S_\alpha \) has boundary \( T \) and thus, comparison with \( S \) yields

\[
\mathbf{M}(S \uplus (X \setminus B(x, s)) + S_\alpha) \geq \frac{1}{2} \mathbf{M}(S \uplus B(x, s) - S_\alpha) \leq \mathbf{M}(S).
\]

If, moreover \( S_\alpha \) is chosen such that \( \mathbf{M}(S_\alpha) \leq C[\mathbf{M}((S, q_x, s))]^\alpha \) then the above estimate implies that

\[
\mathbf{M}(S \uplus B(x, s)) \leq 3 \mathbf{M}(S_\alpha) \leq 3C[\mathbf{M}((S, q_x, s))]^\alpha
\]

for almost every \( s \in (0, \text{dist}(x, \text{spt} T)) \). Setting \( \beta(s) := \|S\|(B(x, s)) \) and again using the slicing theorem we obtain

\[
\beta(s) \leq 3C[\beta'(s)]^\alpha
\]

for almost every \( s \in (0, \text{dist}(x, \text{spt} T)) \). Since, by Theorem 1.2 in [Wen1], \( X \) admits an isoperimetric inequality of power \( \alpha_0 := \frac{k+1}{2} \) for \( \mathbf{I}_k(X) \) it follows that

\[
\|S\|(B(x, s)) \geq \frac{s^{k+1}}{(3C)^{k(k+1)\frac{1}{k+1}}} \quad \text{for all} \ 0 \leq s \leq \text{dist}(x, \text{spt} T)
\]

which proves (4). In particular, we have \( \beta(3C(k+1)) \geq 3C \). If now \( \alpha > 1 \) then it follows from (5) that

\[
\beta(s)^{\frac{\alpha-1}{\alpha}} \geq \beta(3C(k+1))^{\frac{\alpha-1}{\alpha}} + \frac{\alpha-1}{\alpha} \cdot \frac{s - 3C(k+1)}{(3C)^{1/\alpha}}
\]

\[
\geq (3C)^{\frac{\alpha-1}{\alpha}} \left\{ 1 + \frac{\alpha-1}{\alpha} \left[ \frac{s}{3C} - (k+1) \right] \right\}
\]

from which the second statement follows for \( \alpha > 1 \). On the other hand, if \( \alpha = 1 \) then

\[
\beta(s) \leq 3C \beta'(s)
\]

and hence

\[
\frac{s - 3C(k+1)}{3C} \leq \ln \left( \frac{\beta(s)}{\beta(3C(k+1))} \right) \leq \ln \left( \frac{\beta(s)}{3C} \right).
\]

This prove the lemma.

We are now ready to prove the main theorem. The proof of the first part of the theorem is a variation of the arguments given in [LaSc]. The second part uses asymptotic cones. We refer the reader to [KILe] for a general reference.

**Proof of Theorem 1.3.** We begin by proving the first statement of the theorem. For this let \( F \subset X \) be a flat of maximal dimension \( k + 1 = \text{Rank} X \) and fix a point \( x_0 \in X \). By Lemma 2.4 we may assume without loss of generality that \( x_0 \in F \). We show that \( \delta_{x_0, \varrho, A_0}(r) \geq r^{k+2} \) for every \( \varrho \in (0, 1) \) and where \( A_0 := \mathcal{H}^k(S(x_0, 1) \cap F) \). To do so fix \( r > 24k + 1 \) and
where \( s_0 > 0 \) is chosen as below and let \( T := [S(x_0, r) \cap F] \in \mathbf{I}_k(X) \) be the integral current induced by the sphere \( S(x_0, r) \cap F \). It is clear that \( \mathbf{M}(T) = \mathcal{H}^k(S(x_0, r) \cap F) = A_0 r^k \). Let furthermore \( S \in \mathbf{I}_{k+1}(X) \) be such that \( \partial S = T \) and \( \text{spt} S \subset X \setminus U(x_0, qr) \). Denote by \( \pi : X \to F \) the orthogonal projection onto \( F \) and observe that by the constancy theorem \([\text{Fed, 4.1.7}]\)

\[
\pi \# S = [\chi_{B(x_0, r) \cap F}]_k
\]

where the right-hand side denotes the current induced by the \((k+1)\)-dimensional ball of radius \( r \) and center \( x_0 \) in \( F \). Let now \( Q \subset F \cap B(x_0, qr/4) \) be a closed \((k+1)\)-dimensional Euclidean cube of edge length \( 3s_0 \). We adapt the argument in the proof of Proposition 3.2 in \([\text{LaSc}]\) to estimate \( \|S\|(\pi^{-1}(Q)) \) from below. For this set \( \nu := 2^{k+1} \) and let the vertices \( q_1, \ldots, q_\nu \) of \( Q \) be ordered in such a way that each segment \([q_i, q_{i+1}]\), \( i = 1, \ldots, \nu - 1 \), is an edge of \( Q \). Set \( P = \bigcup_{i=1}^{\nu} [q_i, q_{i+1}] \) and denote by \( R \) the union of all \((k+1)\)-cubes of \( Q \) with edge length \( s_0 \) which meet \( P \). Let \( Q_i \subset Q \) be the cube of edge length \( s_0 \) which contains \( q_i \). Denoting by \( Z \) the common \( k \)-face of \( R \) and \( Q_1 \) we define a 1-Lipschitz map \( \psi : R \to Z \) in such a way that each fiber \( \psi^{-1}(\{z\}) \) is a connected polygonal arc lying at constant distance from \( P \). After possibly changing the edge length of \( Q \) by an arbitrarily small amount we may assume by the slicing theorem that \( S \perp \pi^{-1}(Q) \in \mathbf{I}_{k+1}(X) \). Furthermore, \( S_z := \langle S \perp \pi^{-1}(Q), \psi \circ \pi, z \rangle \in \mathbf{I}_1(X) \) for almost every \( z \in Z \). Since

\[
\text{spt} \partial (S \perp \pi^{-1}(Q)) \subset \pi^{-1} \left( \overline{F \setminus Q} \right)
\]

it follows that \( \partial S_z = (-1)^k (\partial (S \perp \pi^{-1}(Q)), \psi \circ \pi, z) \) is supported in \( \pi^{-1} \left( \overline{F \setminus Q} \right) \). Furthermore we have

\[
\pi \# S_z = \langle \pi \# (S \perp \pi^{-1}(Q)), \psi, z \rangle = \langle (\pi \# S) \perp Q, \psi, z \rangle = [\psi^{-1}(\{z\})]
\]

which implies that \( \text{spt} S_z \) has a connected component whose image under \( \pi \) is \( \psi^{-1}(\{z\}) \). Therefore, if \( s_0 \) is chosen suitably large (only depending on \( X \)) it follows from Lemma 3.1 in \([\text{LaSc}]\) that

\[
\mathbf{M}(S_z) \geq \lambda qr / 4
\]

where \( \lambda > 0 \) is a constant only depending on \( X \). (Note that for this we used the fact that \( \text{dist}(\text{spt} S_z, F) \geq qr/4 \).) Application of the slicing theorem finally yields

\[
\|S\|(\pi^{-1}(Q^o)) \geq \int Z \mathbf{M}(S_z)dz \geq \frac{\lambda qr}{4} s_0^k r.
\]

Since \( B(x_0, qr/4) \cap F \) contains at least

\[
4 \left( \frac{qr}{12\sqrt{k+1}s_0} - 1 \right)^{k+1}
\]

cubes of edge length \( 3s_0 \) and whose interiors are pairwise disjoint we obtain

\[
\mathbf{M}(S) \geq C r^{k+2}
\]
for a constant $C$ depending only on $\varrho$, $\lambda$, $s_0$, and $k$. This proves that $\delta^k_{x_0,\varrho,\lambda}(r) \geq r^{k+2}$ and therefore completes the proof of the first statement.

We now turn to the proof of the second part of the theorem. The proof is by contradiction. Assume therefore that $\text{div}_k(X)$ grows faster than $r^{k+1}$. There thus exist for $\varrho := \frac{1}{4}$ an $x_0 \in X$ and $A > 0$ such that

$$\limsup_{r \to \infty} \frac{\delta^k_{x_0,\varrho,\lambda}(r)}{r^{k+1}} = \infty.$$ 

In particular, there exist an increasing sequence $(r_m) \subset [0, \infty)$ with $r_m \to \infty$ as $m \to \infty$ and a sequence $(T_m) \subset I_k(X)$ with $\partial T_m = 0$ and such that the following properties hold:

(i) $\text{spt} T_m \subset S(x_0, r_m)$ compact for all $m \in \mathbb{N}$

(ii) $r_m^{-(k+1)}\text{Fillvol}_{X \setminus U(x_0, r_m/4)}(T_m) \to \infty$ as $m \to \infty$

(iii) $\mathbf{M}(T_m) \leq Ar_m^k$ for every $m \in \mathbb{N}$.

By [Wen1, Theorem 1.6] there exists for every $m \in \mathbb{N}$ an absolutely area-minimizing $S_m \in I_{k+1}(X)$ with $\partial S_m = T_m$. By [Wen1, Theorem 1.2] we have

$$\mathbf{M}(S_m) \leq C[\mathbf{M}(T_m)]^{\frac{k+1}{k+2}} \leq CA^{\frac{k+1}{k+2}} r_m^{k+1}$$

for some constant $C$ depending only on $k$, and in view of property (ii) we may therefore assume that

$$\text{spt} S_m \cap B(x_0, r_m/4) \neq \emptyset$$

for every $m \in \mathbb{N}$. If we fix a point $x_m$ in the intersection then clearly $\text{dist}(x_m, \text{spt} T_m) \geq \frac{3}{4} r_m$. By the slicing theorem and inequality (6) there exists $r'_m \in (\frac{1}{2} r_m, \frac{3}{4} r_m)$ such that $S'_m := S_m \setminus B(x_0, r'_m) \in I_{k+1}(X)$ and

$$\mathbf{M}(\partial S'_m) \leq 4CA^{\frac{k+1}{k+2}} r_m^{k}.$$ 

We set $T'_m := \partial S'_m$ and define a sequence of metric spaces $Y_m := (B(x_0, r_m), \frac{1}{r_m} d_X)$ where $d_X$ denotes the metric on $X$. Note that $\text{diam} Y_m \leq 2$. Setting $Z_m := \text{spt} S'_m \subset Y_m$ it follows directly from the growth estimate in (4) that the sequence $(Z_m, \frac{1}{r_m} d_X)$ is equi-compact and equibounded. Therefore, by Gromov's compactness theorem there exists (after passage to a subsequence) a compact metric space $(Z, d_Z)$ and isometric embeddings $\varphi_m : (Z_m, \frac{1}{r_m} d_X) \hookrightarrow (Z, d_Z)$ and we may assume that $\varphi_m(Z_m)$ is a Cauchy sequence with respect to the Hausdorff distance. Denote by $S'_m$ the current $S'_m$ viewed as an element of $I_{k+1}(Y_m)$. Since $\mathbf{M}(\varphi_m \# S'_m) \leq CA^{(k+1)/k}$ and $\mathbf{M}(\varphi_m \# S'_m) \leq 4CA^{(k+1)/k}$ we may assume by the compactness and closure theorems for currents that $\varphi_m \# S'_m$ weakly converges to some $S \in I_{k+1}(Z)$. We show that $\partial S \neq 0$. For this we define the metric space $Y$ as the disjoint union $\bigsqcup_{m=1}^{\infty} Y_m$ and endow it with the metric $d_Y$ such that $d_Y|_{Y_m \times Y_m} = \frac{1}{r_m} d_X$ as well as $d_Y(y, y') = 3$ whenever $y \in Y_m$ and $y' \in Y_{\ell}$ with $\ell \neq m$. It is clear that $Y$ is 2-quasi-convex and admits a local cone type inequality for $I_n(Y)$, $n = 1, \ldots, k$, in the sense
of [Wen2]. Denote by $T'_m$ the current $T'_m$ viewed as an element of $I_k(Y)$ and note that $M(T''_m) \leq 4CA^k$. We show that $T''_m$ does not weakly converge to 0. Assume $T''_m$ weakly converges to 0. Then by Theorem 1.4 in [Wen2] we have Fillvol($T''_m$) $\to 0$. In particular, there then exist absolute area-minimizers $\hat{S}_m \in I_{k+1}(Y)$ with $\partial \hat{S}_m = T''_m$ for all $m \in \mathbb{N}$ and such that $M(\hat{S}_m) = 0$. Denote by $\tilde{S}_m$ the current $\hat{S}_m$ viewed as an integral current in $X$. Then $\tilde{S}_m$ is absolutely area-minimizing and satisfies $\partial \tilde{S}_m = T'_m$ and

$$\frac{M(\tilde{S}_m)}{r^{k+1}_{m+1}} = \frac{M(\hat{S}_m)}{r^{k+1}_{m+1}} \to 0.$$ 

It then follows directly from the growth estimate (4) that

$$\text{spt} \tilde{S}_m \subset X \setminus U(x_0, r_m/4)$$

for $m$ large enough. This leads to a contradiction with (ii). Indeed, $S_m - S'_m + \tilde{S}_m$ is a filling of $T_m$ with support outside $U(x_0, r_m/4)$ and with mass bounded from above by $D_1 + 1$ for a suitable constant $D$. This shows that $T''_m$ does not weakly converge to 0 and therefore there exist Lipschitz maps $f, \pi_1, \ldots, \pi_k \in \text{Lip}(Y)$ and $\varepsilon > 0$ such that

$$T''_m(f, \pi_1, \ldots, \pi_k) \geq \varepsilon \quad \text{for all} \quad m \in \mathbb{N}.$$ 

Note that since $Y$ is a bounded metric space the functions $f$ and $\pi_i$ are bounded. We define Lipschitz functions $\hat{f}_m$ and $\hat{\pi}^m_i$ on $\varphi_m(Z_m)$ by $\hat{f}_m(z) := f(\varphi_m^{-1}(z))$ and $\hat{\pi}^m(z) := \pi_i(\varphi_m^{-1}(z))$ for $z \in \varphi_m(Z_m)$. Here we view $\varphi_m^{-1}$ as a map from $\varphi(Z_m)$ to $Y = \bigcup_{\ell=1}^{\infty} Y_\ell$ with image in $Y_m \subset Y$. By McShane’s extension theorem there exist extensions $\hat{f}_m, \hat{\pi}^m_i : Z \to \mathbb{R}$ of $f_m$ and $\pi^m_i$ with the same Lipschitz constants as $f$ and $\pi_i$. By Arzelà-Ascoli theorem we may assume that $f_m$ and $\pi^m_i$ converge uniformly to Lipschitz maps $\hat{f}, \hat{\pi}_i$ on $Z$. Finally we abbreviate $T''_m := \varphi_m \# T''_m$ and use [AmKi1, Proposition 5.1] to estimate

$$\partial S(\hat{f}, \hat{\pi}_1, \ldots, \hat{\pi}_k) = \lim_{m \to \infty} T''_m(\hat{f}, \hat{\pi}_1, \ldots, \hat{\pi}_k) = \lim_{m \to \infty} \left[ T''_m(\hat{f}_m, \hat{\pi}^m_1, \ldots, \hat{\pi}^m_k) + T''_m(\hat{f} - \hat{f}_m, \hat{\pi}_1, \ldots, \hat{\pi}_k) \right. \right. \left. \left. + T''_m(\hat{f}_m, \hat{\pi}_1, \ldots, \hat{\pi}_k) - T''_m(\hat{f}_m, \hat{\pi}^m_1, \ldots, \hat{\pi}^m_k) \right] \right.$$

$$\geq \varepsilon - \lim\sup_{m \to \infty} \left[ \prod_{i=1}^{k} \text{Lip}(\hat{\pi}_i) \int_{Z} |\hat{f} - \hat{f}_m|d\|T''_m\| \right]$$

$$- \lim\sup_{m \to \infty} \left[ \text{Lip}(\hat{f}_m) \sum_{i=1}^{k} \int_{Z} |\hat{\pi}_i - \hat{\pi}^m_i|d\|T''_m\| \right]$$

$$= \varepsilon.$$

This shows that indeed $\partial S \neq 0$ and hence also $S \neq 0$. Finally fix an ultrafilter $\omega$ on $\mathbb{N}$ and denote by $X_\omega$ the asymptotic cone of the sequence $(X, \frac{1}{r_m}d_X, x_0)$. We construct a map $\psi : Z' \to X_\omega$ where $Z' := \text{lim}_H \varphi_m(Z_m) \subset Z$ is the limit
with respect to the Hausdorff distance. For \( z \in Z' \) there exists \( z_m \in Z_m \) such that \( \varphi_m(z_m) \to z \). We set \( \psi(z) := (z_m)_{m \in \mathbb{N}} \). It is straightforward to check that \( \psi \) is well-defined and an isometric embedding. Since \( \text{spt} S \subset Z' \) we obtain that \( \psi_S \) is a non-zero \((k+1)\)-dimensional integral current in \( X_\omega \).

By Theorem 4.5 (parametric representation) in [AmKi1] there then exists a bi-Lipschitz map \( \nu : K \subset \mathbb{R}^{k+1} \to X_\omega \) where \( K \) is measurable and of strictly positive Lebesgue measure. We can then use [Kir] and Theorem A of [Kle] to conclude that the geometric dimension of \( X_\omega \) is at least \( k+1 \) and hence, by Theorem C of [Kle], that the Euclidean rank of \( X \) is at least \( k+1 \) which contradicts our assumption that \( \text{Rank} X \leq k \). □

Remark 3.2. We point out that for the second part of the statement properness of \( X \) is only used for Theorem C of [Kle].

We mention that if \( X \) is a cocompact Hadamard manifold we need not use asymptotic cones to prove the second part of the theorem. We can use (the proof of) Theorem 1 of [AnSc] in the following way instead. Let \( S_m \) and \( x_m \) be as in the proof above. By cocompactness and the compactness theorem for integral currents we may assume without loss of generality that \( x_m \) converges to some \( y \in X \) and that \( S_m \) converges to an absolutely area-minimizing local integral current \( \Sigma \in \mathcal{I}^{k+1}_{k+1}(X) \) with \( \partial \Sigma = 0 \). Using the monotonicity formula, inequality (6) and the fact that \( \text{dist}(x_m, \text{spt} \partial S_m) \geq \frac{3r_m}{4} \) one readily obtains

\[
\omega_{k+1} r^{k+1} \leq \| \Sigma (B(y, r)) \| \leq Cr^{k+1}
\]

for some constant \( C \) and for all \( r > 0 \). Then the proof of Theorem 1 in [AnSc] (see (3.3) and thereafter) implies the existence of a \((k+1)\)-dimensional flat in \( X \) which contradicts the assumption that \( \text{Rank} X \leq k \).

4. Divergence versus isoperimetric inequality

The purpose of this section is to prove Proposition 1.8 as well as to establish the linear isoperimetric inequality Theorem 1.7 needed to conclude Corollary 1.9. We first prove Proposition 1.8.

Proof of Proposition 1.8. Fix \( x_0 \in X \) and set \( \varrho_0 := \frac{1}{2} \). Let \( A > 0 \) be arbitrary and let \( r > 0 \) be large enough (as chosen below). For a \( T \in \mathcal{I}_k(X) \) with \( \text{spt} T \subset S(x_0, r) \), \( \partial T = 0 \) and \( \mathbf{M}(T) \leq Ar^k \) let \( S \in \mathcal{I}_k+1(X) \) be as in Lemma 3.1. We show that \( S \) is an admissible filling in the sense that its support is contained in \( X \setminus U(x_0, r/2) \). Indeed, we have

\[
\mathbf{M}(S) \leq C[\mathbf{M}(T)]^\alpha \leq CA^\alpha r^{k\alpha}
\]

which together with the growth estimate in Lemma 3.1 yields for \( x \in \text{spt} S \)

\[
\text{dist}(x, \text{spt} T) \leq \begin{cases} D_1 + D_2 r^{k(\alpha-1)} & \text{if } \alpha > 1 \\ D_1 + D_2 \ln r & \text{if } \alpha = 1 \end{cases}
\]
Theorem 4.1. Let $M$ be a geodesic parameterized by arc-length on $\mathcal{X}$, where $\kappa > 0$. In particular, if $r$ for a geodesic from $x$ to $z$ we set $\varphi(t,x) = c_{x,z}(t)$ where $c_{x,z} : [0,1] \to \mathcal{X}$ denotes the constant speed geodesic from $x_0$ to $x$. We furthermore define

$$s_\kappa(r) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sinh(\sqrt{\kappa}r) & \text{if } \kappa < 0 \\ r & \text{if } \kappa = 0 \\ \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}r) & \text{if } \kappa > 0 \end{cases}$$

for $r \geq 0$. We note that $s_\kappa$ is the norm of a normal Jacobi field $Y$ along a geodesic parameterized by arc-length on $\mathcal{M}_k^2$ with $Y(0) = 0$ and $\|Y'(0)\| = 1$.

**Theorem 4.1.** Let $(\mathcal{X}, d)$ be a complete CAT($\kappa$)-space and $x_0 \in \mathcal{X}$. Let $k \geq 1$ and suppose $T \in \mathbf{I}_k(\mathcal{X})$ has support in the ball $B(x_0, R)$, where $R < \frac{1}{2}D_\kappa$ in case $\kappa > 0$. Then $x_0 \# T := \varphi_\#([0,1] \times T)$ satisfies

$$\mathbf{M}(x_0 \# T) \leq \int_{B(x_0, R)} \frac{d(x_0,x)}{|s_\kappa(d(x_0,x))|^k} \left( \int_0^1 |s_\kappa(td(x_0,x))|^k dt \right) d\|T\|(x).$$

In particular, if $\kappa = 0$ then we obtain

$$\mathbf{M}(x_0 \# T) \leq \frac{R}{k+1} \mathbf{M}(T).$$

**Remark 4.2.** For Banach spaces (even finite dimensional) inequality (7) is in general false as Example 10.3 in [AmKi1] illustrates. The main reason for this is the additional area factor appearing in the mass.

**Lemma 4.3.** Let $(\mathcal{X}, d)$ be an Alexandrov space of curvature bounded above and $\varphi : K \to \mathcal{X}$ a Lipschitz map where $K \subset \mathbb{R}^k$ is measurable. Suppose $z \in K$ is a density point, $\mathbf{md} \varphi_z$ exists, is non-degenerate and satisfies (1). Then the metric differential $\mathbf{md} \varphi_z$ is induced by an inner product.

**Proof.** Let $z \in K$ be as in the assumption. We show that $(\mathbb{R}^k, \mathbf{md} \varphi_z)$ is an inner product space. To do so we prove that for any four vectors $v_1, \ldots, v_4 \in \mathbb{R}^k$ there exists a subembedding in $\mathbb{R}^2$, i.e. there exist four points $w_1, \ldots, w_4 \in \mathbb{R}^2$ such that $\mathbf{md} \varphi_z(v_i - v_{i+1}) = |w_i - w_{i+1}|$, $i \in \mathbb{N}$ mod 4, as
well as $\text{md} \varphi_z(v_1 - v_3) \leq |w_1 - w_3|$ and $\text{md} \varphi_z(v_2 - v_4) \leq |w_2 - w_4|$. As mentioned in Section 2 this will then imply that $(\mathbb{R}^k, \text{md} \varphi_z)$ is a CAT(0)-space. Since every normed space which is CAT(0) is in fact a pre-Hilbert space this will prove the lemma.

To prove the existence of a subembedding let $\varepsilon > 0$ be such that $B := B(\varphi(x), \varepsilon)$ is CAT($\kappa$) for some $\kappa \in \mathbb{R}$. Then $B$ endowed with the rescaled metric $d_r := \frac{1}{r}d$ is CAT($\kappa\sqrt{r}$) for every $r > 0$. We first assume $\kappa \leq 0$ so that $(B, d_r)$ is CAT(0). Since $z$ is a Lebesgue density point of $K$ we may assume after approximation that $z + rv_i \in K$ for $r > 0$ small enough and for $i = 1, \ldots, 4$. For $r > 0$ sufficiently small there thus exist points $w^*_1, \ldots, w^*_n \in \mathbb{R}^2$ such that

- $\frac{1}{r}d(\varphi(z + rv_i), \varphi(z + rv_{i+1})) = |w^*_i - w^*_i|$ for $i \in \mathbb{N}$ mod 4
- $\frac{1}{r}d(\varphi(z + rv_1), \varphi(z + rv_3)) \leq |w^*_1 - w^*_3|$
- $\frac{1}{r}d(\varphi(z + rv_2), \varphi(z + rv_4)) \leq |w^*_2 - w^*_4|$

Of course it is not restrictive to assume that $w^*_i = 0$ for all $r > 0$. By the Lipschitz continuity of $\varphi$ we conclude that all $w^*_i$ lie in a fixed ball centered at 0. There then exists a sequence $r_n$ converging to 0 such that $w^*_i$ converges to some $w_i$ as $n \to \infty$ for every $i$. It then follows immediately from property (1) that $w_1, \ldots, w_4$ constitute a subembedding of $v_1, \ldots, v_4$. This concludes the proof in the case $\kappa \leq 0$. The case $\kappa > 0$ is almost analogous. It is enough to note that the comparison spaces $\mathcal{M}^2_{\kappa\sqrt{r}}$ in which the comparison points $w^*_i$ lie converge to $\mathbb{R}^2$ in the pointed Hausdorff-Gromov metric (the base point being the north pole) as $r \searrow 0$.

Proof. By Theorem 4.5 in [AmKi1] it is not restrictive to assume that $T = \psi_{\#}[\theta]$ for some bi-Lipschitz map $\psi : K \to X$, $K \subset \mathbb{R}^k$ compact, and $\theta \in L^1(K, \mathbb{Z})$. We give an explicit formula for $x_0 \ast T$. For this, let $(f, \pi_1, \ldots, \pi_k) \in \mathcal{D}^k(X)$. We define $g_t := g \circ \varphi(t, \cdot)$ whenever $g \in \text{Lip}(X)$ and furthermore write $\tilde{\pi}^i := (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_k)$. We compute

\[
(x_0 \ast T)(f, \pi_1, \ldots, \pi_{k+1}) = (\int_{[0, 1]} T(f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_k \circ \varphi)) = \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 T\left(f_t \frac{\partial \pi_t}{\partial t}, \pi_1 t, \ldots, \pi_{i-1} t, \pi_{i+1} t, \ldots, \pi_{k+1} t\right) dt
\]

\[
= \sum_{i=1}^{k+1} (-1)^{i+1} \int_0^1 \int_K \theta f \circ \tilde{\varphi} \frac{\partial (\pi_t \circ \varphi)}{\partial t} \det(\nabla_{R^k}(\tilde{\pi}^i \circ \tilde{\varphi})) d\mathcal{L}^k dt
\]

\[
= \int_{[0, 1] \times K} \tilde{\theta} f \circ \tilde{\varphi} \det(\nabla (\pi \circ \varphi)) d\mathcal{L}^{k+1}
\]

\[
= (\tilde{\varphi}_{\#}[\theta])(f, \pi_1, \ldots, \pi_{k+1}),
\]
where we have furthermore set \( \tilde{\varphi}(t, z) := \varphi(t, \psi(z)) \) and \( \tilde{\theta}(t, z) := \theta(z) \). Since

\[
| \det \left( \frac{\partial (\pi_i \circ \varphi)}{\partial x_j} (t, z) \right) | \leq \lambda_{\text{md} \tilde{\varphi}(t, z)} J_{k+1}(\text{md} \tilde{\varphi}(t, z))
\]

we easily obtain using the definition of mass and the area formula for Lipschitz maps [Kir] that

\[
(8) \quad \mathbf{M}(\tilde{\varphi}(\#)) \leq \int_A |\theta(z)| \lambda_{\text{md} \tilde{\varphi}(t, z)} J_{k+1}(\text{md} \tilde{\varphi}(t, z)) d\mathcal{L}^{k+1}(t, z).
\]

where \( A \) is the set of points \((t, z) \in [0, 1] \times K\) such that \( z \) is a Lebesgue point of \( K \) and \( \text{md} \tilde{\varphi}(t, z) \) exists, is non-degenerate, and satisfies (1). By Lemma 4.3 above \( \text{md} \tilde{\varphi}(t, z) \) comes from an inner product for all \((t, z) \in A\) and therefore \( \lambda_{\text{md} \tilde{\varphi}(t, z)} = 1 \). We now estimate \( J_{k+1}(\text{md} \tilde{\varphi}(t, z)) \). We may assume without loss of generality that \( \text{md} \psi_z \) exists for all \((t, z) \in A\) and is induced by an inner product and that the function \( \tilde{\nu}(z') := d(x_0, \psi(z')) \) is differentiable at the point \( z \). Choose an orthonormal basis \( \{v_1, \ldots, v_k\} \) of \( \mathbb{R}^k \) endowed with the inner product inducing \( \text{md} \psi_z \) in such a way that \( v_1 \) is parallel to \( \nabla \tilde{\nu}(z) \) and set

\[
Q := \left\{ \left( s, \sum_{i=1}^k r_i v_i \right) : 0 \leq s \leq 1, 0 \leq r_i \leq 1 \right\} \subset \mathbb{R} \times \mathbb{R}^k.
\]

If \( \nabla \tilde{\nu}(z) = 0 \) then we do not pose any restriction on the choice of \( v_1 \). It follows that

\[
J_{k+1}(\text{md} \tilde{\varphi}(t, z)) = \frac{\mathcal{H}_{\text{md} \tilde{\varphi}(t, z)}^{k+1}(Q)}{\mathcal{H}_{\text{eucl}}^{k+1}(Q)} = \frac{\mathcal{H}_{\text{md} \tilde{\varphi}(t, z)}^{k+1}(Q)}{\mathcal{H}_{\text{md} \tilde{\varphi}(t, z)}^{k+1}(Q)} J_k(\text{md} \psi_z)
\]

and

\[
\mathcal{H}_{\text{md} \tilde{\varphi}(t, z)}^{k+1}(Q) \leq \text{md} \tilde{\varphi}(t, z)(1, 0) \text{md} \tilde{\varphi}(t, z)(h, v_1) \prod_{i=2}^k \text{md} \tilde{\varphi}(t, z)(0, v_i)
\]

for every \( h \in \mathbb{R} \). The latter is a consequence of the fact that \( \text{md} \tilde{\varphi}(t, z) \) comes from an inner product.

We first estimate \( \text{md} \tilde{\varphi}(t, z)(0, v_i) \) from above for \( i \geq 2 \). Fix an \( i \geq 2 \) and set \( v := v_i \). For \( r > 0 \) small enough let \( \overline{\Delta}(x_0, \overline{x}, \overline{x}_r) \) be the comparison triangle in \( \mathcal{M}^2_\kappa \) of the triangle \( \Delta(x_0, \psi(z), \psi(z + rv)) \). By a simple approximation argument we may assume that \( z + rv \in K \) for all \( r > 0 \) small enough. Denoting by \( \alpha_r \) the angle at \( \overline{x} \) between the geodesics \([\overline{x}, \overline{x}_0]\) and \([\overline{x}, \overline{x}_r]\) one verifies, using the law of cosines for \( \mathcal{M}^2_\kappa \), see [BrHa, I.2.13], that \( \alpha := \lim_{r \searrow 0} \alpha_r \) exists and satisfies

\[
\cos \alpha = -\frac{d\tilde{\nu}_z(v)}{\text{md} \psi_z(v)} = 0
\]
and hence \( \alpha = \frac{\pi}{2} \). Denote by \( \tilde{\tau}(\cdot, r) \) the geodesic parameterized on \([0, 1]\) from \( x_0 \) to \( x_r \). Then, by the CAT(\( \kappa \))-condition we obtain

\[
\mathrm{md} \tilde{\varphi}_{(t, z)}(0, v) \leq \limsup_{r \searrow 0} \frac{1}{r} \delta_{\mathcal{M}^2}(\tau(t, 0), \tau(t, r)) = \Vert Y(t) \Vert,
\]

where \( Y \) is the normal Jacobi field along \( \tau(\cdot, 0) \) with \( Y(0) = 0 \) and \( \Vert Y(1) \Vert = \mathrm{md} \psi_z(v) = 1 \), and consequently

\[
\mathrm{md} \tilde{\varphi}_{(t, z)}(0, v_i) \leq \frac{s_{\kappa}(td(x_0, \psi(z)))}{s_{\kappa}(d(x_0, \psi(z)))}
\]

for every \( i \in \{2, 3, \ldots, k\} \).

Next, we estimate \( \mathrm{md} \tilde{\varphi}_{(t, z)}(h, v_1) \) for a suitable \( h \in \mathbb{R} \) by proceeding in a similar way. Let \( \Delta(x_0, x_r) \), \( \alpha_r \), and \( r \) be as above, but with \( v := v_1 \). After possibly replacing \( v_1 \) by \(-v_1\) we may assume that \( \alpha_r \geq \frac{\pi}{2} \). Let \( h(r) \in \mathbb{R} \) be such that the triangle \( \Delta(x_0, x_\tau(1 + rt^{-1}h(r), r)) \) in \( \mathcal{M}^2 \) has a right angle at the vertex \( x_r \). Then one easily checks that

\[
h(r) \to h := -\frac{tdv_z(v_1)}{\nu(z)} \leq 0 \quad \text{as } r \searrow 0
\]

and, since \( d(x_0, x_r) \leq R < \frac{1}{2} D_{\kappa} \),

\[
\limsup_{r \searrow 0} \frac{1}{r} \delta_{\mathcal{M}^2}(x, \tau(1 + rt^{-1}h(r), r)) \leq \limsup_{r \searrow 0} \frac{1}{r} \delta_{\mathcal{M}^2}(x, x_r)
\]

and hence

(9) \[
\limsup_{r \searrow 0} \frac{1}{r} \delta_{\mathcal{M}^2}(x, \tau(1 + rt^{-1}h(r), r)) \leq \mathrm{md} \psi_z(v_1) = 1.
\]

The CAT(\( \kappa \))-condition then implies

\[
\mathrm{md} \tilde{\varphi}_{(t, z)}(h, v_1) = \lim_{r \searrow 0} \frac{1}{r} d(\tilde{\varphi}(t + rh, z + rv_1), \tilde{\varphi}(t, z))
\]

\[
= \lim_{r \searrow 0} \frac{1}{r} d(\tilde{\varphi}(t(1 + rt^{-1}h(r)), z + rv_1), \tilde{\varphi}(t, z))
\]

\[
\leq \limsup_{r \searrow 0} \frac{1}{r} \delta_{\mathcal{M}^2}(\tau(t(1 + rt^{-1}h(r)), r), \tau(t, 0))
\]

\[
= \Vert Y(t) \Vert,
\]

where \( Y \) denotes the normal Jacobi field along \( \tau(\cdot, 0) \) with \( Y(0) = 0 \) and

\[
\Vert Y(1) \Vert = \limsup_{r \searrow 0} \frac{1}{r} \delta_{\mathcal{M}^2}(\tau(1 + rt^{-1}h(r), r), x_r).
\]

Together with (9) we obtain

\[
\mathrm{md} \tilde{\varphi}_{(t, z)}(h, v_1) \leq \frac{s_{\kappa}(td(x_0, \psi(z)))}{s_{\kappa}(d(x_0, \psi(z)))}.
\]
Since \( \tilde{d}(t,z)(1,0) = d(x_0, \psi(z)) \) we finally see that
\[
\mathbf{J}_{k+1}(\tilde{d}(t,z)) \leq d(x_0, \psi(z))^{\left\lfloor \frac{s_k(td(x_0, \psi(z)))}{s_k(d(x_0, \psi(z)))} \right\rfloor} \mathbf{J}_k(\tilde{d}(x_0, \psi(z)))
\]
which, together with (8), proves the theorem. \( \square \)

The cone type inequality with Euclidean constant can be used to prove the following monotonicity formula for absolutely area minimizing currents which is well-known for integral currents in \( \mathbb{R}^n \).

**Corollary 4.4.** Let \((X, d)\) be a Hadamard space, \(k \geq 1\), and let \(S \in I_k(X)\) be absolutely area minimizing. If \(x_0 \in \text{spt}(S) \setminus \text{spt}(\partial S)\) then the function
\[
f : (0, \infty) \to [0, \infty) \quad r \mapsto \frac{\|S\|(B(x_0, r))}{\omega_k r^k}
\]
is monotonically non-decreasing on \([0, \text{dist}(x_0, \text{spt}(\partial S))]\).

It should be mentioned that for arbitrary \(S \in I_k(X)\)
\[
\lim_{r \to 0} \frac{\|S\|(B(x_0, r))}{\omega_k r^k} \geq 1
\]
for \(|S|\)-almost every \(x_0 \in \text{spt} S\). This follows from [Kir] together with the representation formula for the mass (Theorem 9.5 in [AmKi1]) and Lemma 4.3 (which implies that the area factor equals 1).

**Proof.** Denote by \(\varrho\) the distance function to the point \(x_0\) and define \(\beta(r) := \|S\|(B(x_0, r))\). By the slicing theorem we have that \(\partial(S \sqcup B(x_0, r)) = \langle S, \varrho, r \rangle \in I_{k-1}(X)\) and
\[
M(\langle S, \varrho, r \rangle) \leq \beta(r)
\]
for almost every \(r \in [0, \text{dist}(x_0, \text{spt}(\partial S))]\). Since \(S\) is absolutely area minimizing we furthermore have by Theorem 4.1
\[
\beta(r) = M(S \sqcup B(x_0, r)) \leq M(x_0 \neq \langle S, \varrho, r \rangle) \leq \frac{r}{k+1} \beta'(r)
\]
and consequently \(\frac{k+1}{r} \leq \frac{d}{dt} \log(\beta(r))\) for a.e. \(0 < r < \text{dist}(x_0, \text{spt}(\partial S))\). The claim now follows by integration. \( \square \)

We finally prove that every complete CAT(\(\kappa\))-space \(X\) with \(\kappa < 0\) admits a linear isoperimetric inequality for \(I_k(X)\), \(k \geq 1\).

**Proof of Theorem 1.7.** We fix an arbitrary \(x_0 \in X\) and define \(S := x_0 \neq T\) as in Theorem 4.1. Note that \(S \in I_{k+1}(X)\) and \(\partial S = T\). For \(0 < r < \infty\) we have
\[
\int_0^1 \sinh^k(\sqrt{-\kappa}r) dt \leq \frac{1}{k r^\sqrt{-\kappa}} \sinh^k(\sqrt{-\kappa}r)
\]
which, together with Theorem 4.1, implies
\[
M(S) \leq \frac{1}{\sqrt{-\kappa} k} M(T),
\]
independently of the choice of \(x_0\). \( \square \)
FILLING INVARIANTS IN HADAMARD SPACES

REFERENCES


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