ISOPERIMETRIC INEQUALITIES AND THE ASYMPTOTIC RANK OF METRIC SPACES

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A . In this article we study connections between the asymptotic rank of a metric space and higher-dimensional isoperimetric inequalities. We work in the class of metric spaces admitting cone type inequalities which, in particular, includes all Hadamard spaces, i. e. simply connected metric spaces of non-positive curvature in the sense of Alexandrov. As was shown by Gromov, spaces with cone type inequalities admit isoperimetric inequalities of at most Euclidean type. Here we prove that they admit isoperimetric inequalities of sub-Euclidean type for k-cycles whenever k is greater or equal to their asymptotic rank. As a consequence it follows that the higher-dimensional isoperimetric inequalities can be used to detect the asymptotic rank of such spaces. Our work is to some extent inspired by a conjecture of Gromov which, in the case of proper cocompact Hadamard spaces, asserts even linear isoperimetric inequalities above the asymptotic rank. Our methods can moreover be used to establish polynomial isoperimetric inequalities for metric spaces admitting polynomial cone type inequalities. These include spaces with polynomial Lipschitz combings.

1. I

Given a complete metric space X and $k \in \mathbb{N}$ the filling volume function FV_{k+1} on X is defined as

$$\text{FV}_{k+1}(r) := \sup\{\text{Fillvol}(T) : T \text{ is a } k\text{-dim. cycle in } X \text{ with } \text{Vol}(T) \leq r\},$$

where $\text{Fillvol}(T)$ is the least volume of a $(k+1)$-chain with boundary T. A suitable notion of k-dimensional chains and cycles in the generality of metric spaces is given by k-dimensional integral currents introduced by Ambrosio and Kirchheim in [3]. Other suitable notions of chains are for example the singular Lipschitz chains of Gromov [14] or, in a simplicial setting, simplicial chains.

Throughout this paper we work with the theory of k-dimensional integral currents in X, the space of which is denoted by $\mathbf{I}_k(X)$. An element $T \in \mathbf{I}_k(X)$ can be thought of as an oriented k-dimensional surface in X (with arbitrary genus and possibly with integer multiplicity) which is locally parametrized by biLipschitz maps from $\mathbb{R}^k$ and whose boundary has finite volume. The volume of an element $T \in \mathbf{I}_k(X)$ is a particular Finsler volume and is called the mass of T and written $\mathbf{M}(T)$, the boundary of T is denoted by ∂T and is an element of $\mathbf{I}_{k-1}(X)$ if $k \geq 1$. The support

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of T is denoted by $\operatorname{spt} T$. The filling volume of a cycle $T \in \mathbf{I}_k(X)$ is then by definition the smallest mass of an $S \in \mathbf{I}_{k+1}(X)$ with $\partial S = T$ and is denoted by $\operatorname{Fillvol}(T)$. We refer to Section 2 for further notation and for remarks concerning relations with other notions of chains.
The following definition of cone type inequality for k-cycles goes back to Gromov [14].

Definition 1.1. *A complete metric space X is said to admit a cone type inequality in dimension k (or a cone type inequality for $\mathbf{I}_k(X)$) if there exists $C > 0$ such that*

$$\operatorname{Fillvol}(T) \leq C \operatorname{diam}(\operatorname{spt} T)\mathbf{M}(T)$$

for every $T \in \mathbf{I}_k(X)$ with $\partial T = 0$ and with bounded support.

Examples are given by Riemannian manifolds without focal points and metric spaces admitting a convex bicombing [28], in particular all simply connected geodesic metric spaces of non-positive curvature in the sense of Alexandrov, called Hadamard spaces, and geodesic metric spaces with convex metric. In [14] Gromov proved that a complete Riemannian manifold which admits cone type inequalities in dimensions $m = 1, \dots, k$ admits an isoperimetric inequality of Euclidean type for k-cycles, thus

$$\operatorname{FV}_{k+1}(r) \leq D r^{\frac{k+1}{k}}$$

for all $r \geq 0$ and for some constant D. In [28] this was shown to hold in the generality of complete metric spaces admitting cone type inequalities. In particular, it follows that Hadamard spaces admit isoperimetric inequalities of Euclidean type in all dimensions $k \geq 1$.
The primary purpose of the present article is to show that the isoperimetric behavior of a metric space changes from Euclidean to sub-Euclidean type in the dimension of the asymptotic rank.

Definition 1.2. *The asymptotic rank of a metric space X, denoted by $\operatorname{asrk}(X)$, is defined as the supremum over $n \in \mathbb{N}$ for which there exists an asymptotic cone X_ω of X and a biLipschitz map $\varphi : K \to X_\omega$ with $K \subset \mathbb{R}^n$ compact and $\mathcal{L}^n(K) > 0$.*

If X is a Hadamard space $\operatorname{asrk}(X)$ is the maximal geometric dimension of an asymptotic cone of X. In particular, if X is a proper and cocompact Hadamard space $\operatorname{asrk}(X)$ coincides with its Euclidean rank, that is the maximal $n \in \mathbb{N}$ for which $\mathbb{R}^n$ isometrically embeds into X. This follows from work of Kleiner [19]. We refer to Section 2 for details and further relations.
Our main result can now be stated as follows.

Theorem 1.3. *Let $k \in \mathbb{N}$ and let X be a complete quasiconvex metric space which admits cone type inequalities for $\mathbf{I}_m(X)$ for $m = 1, \dots, k$. If $k \geq \operatorname{asrk}(X)$ then*

$$\limsup_{r \to \infty} \frac{\operatorname{FV}_{k+1}(r)}{r^{\frac{k+1}{k}}} = 0, \tag{1}$$

thus X admits a sub-Euclidean isoperimetric inequality for $\mathbf{I}_k(X)$.

A metric space X is said to be Q-quasiconvex if for every two points $x, y \in X$ there exists a curve of length at most $Qd(x, y)$ joining x and y; furthermore X is called quasiconvex if it is Q-quasiconvex for some Q.

Theorem 1.3 seems to be the first result in its direction even in the context of (cocompact) Hadamard manifolds. A similar (but stronger) result is known to hold only for symmetric spaces of non-compact type, see below. Theorem 1.3 has interesting applications towards the asymptotic geometry of non-positively curved spaces. Such have recently been exhibited by Kleiner and Lang [20].

In the particular case that $k = \operatorname{asrk}(X) = 1$ our theorem asserts a sub-quadratic isoperimetric inequality for 1-cycles. This is well-known to imply even a linear isoperimetric inequality for 1-cycles, see [13]. It is believed that under suitable conditions (such as for example admitting cone type inequalities) a sub-Euclidean isoperimetric inequality for k-cycles should imply a linear isoperimetric inequality for k-cycles also when $k \geq 2$. Theorem 1.3 might thus be regarded as a first step towards the conjecture of Gromov (somewhat implicitly contained in [15]) which asserts that a proper cocompact Hadamard space X admits linear isoperimetric inequalities for $\mathbf{I}_k(X)$ for all $k \geq \operatorname{asrk}(X)$. In the case $\operatorname{asrk}(X) = 1$ this is known to be true and can be proved via the embedding theorem in [4] and the Lipschitz extension results in [24], see also [23] for a proof in a simplicial setting. As regards the case $\operatorname{asrk}(X) > 1$, the conjecture is known to hold for symmetric spaces of non-compact type but remains open for most other cases, even in the context of Hadamard manifolds.

Below the asymptotic rank the isoperimetric behavior is Euclidean as follows from the next theorem.

Theorem 1.4. *Let X be a complete quasiconvex metric space and let $k \in \mathbb{N}$. Suppose X admits isoperimetric inequalities of Euclidean type for $\mathbf{I}_m(X)$ with some constants D_m, $m = 1, \dots, k-1$. If $k < \operatorname{asrk}(X)$ then*

$$\operatorname{FV}_{k+1}(r) \geq \operatorname{FV}_{k+1}(X, L^\infty(X), r) \geq \varepsilon_k r^{\frac{k+1}{k}}$$

for all $r \geq 0$ and for some $\varepsilon_k > 0$ depending only on D_m, $m = 1, \dots, k-1$.

Here $\operatorname{FV}_{k+1}(X, L^\infty(X), r)$ is defined analogously to $\operatorname{FV}_{k+1}(r)$, with the difference that $\operatorname{Fillvol}(T)$ is replaced by $\operatorname{Fillvol}_{L^\infty(X)}(T)$, the filling volume in $L^\infty(X)$ of $T \in \mathbf{I}_k(X)$. See Section 2.1 for the precise definition. Theorem 1.4 will be a consequence of the stronger Theorem 7.1, which asserts an analogous lower bound for the filling radius function. As regards the constants ε_k in the theorem, it can be shown that a geodesic metric space X with $\operatorname{asrk}(X) > 1$ satisfies

$$\limsup_{r \to \infty} \frac{\operatorname{FV}_2(X, L^\infty(X), r)}{r^2} \geq \frac{1}{4\pi}. \tag{2}$$

In [31] it is moreover proved that a geodesic metric space X which admits a coarse quadratic isoperimetric inequality for curves and for which (2) fails is Gromov hyperbolic and thus all its asymptotic cones are real trees.

It is not known whether Theorem 1.3 holds with the weaker assumption that X merely admits isoperimetric inequalities of Euclidean type. A partial result in this direction is given in Theorem 6.3. Finally, we mention the following consequence of Theorems 1.3 and 1.4.

Corollary 1.5. *Isoperimetric inequalities detect the asymptotic rank of complete quasiconvex metric spaces admitting cone type inequalities.*

The corollary in particular applies to Hadamard spaces. Analogous results for the higher-dimensional divergence invariants of Brady and Farb were obtained in [7], [25], [17] for symmetric spaces of non-compact type and in [30] for proper cocompact Hadamard spaces. These divergence invariants can be thought of as isoperimetric functions at infinity and are loosely related to the $\mathrm{FV}_k(r)$, see [30].
In the following paragraph, we briefly describe the main ideas leading to the proof of our main result. In the subsequent paragraph we will describe additional results which can be proved using the methods developed in the proof of the main theorem.

1.1. **Outline of the proof of Theorem 1.3.** The proof is by contradiction and the aim is to show that, if (1) fails, there exists a sequence of subsets $Z_m \subset X$ and numbers $r_m \to \infty$ such that $(Z_m, \frac{1}{r_m} d_X)$ converges in the Gromov-Hausdorff sense to a compact metric space Z which contains the biLipschitz image of a compact subset $K \subset \mathbb{R}^{k+1}$ for which $\mathcal{L}^{k+1}(K) > 0$. One of the principle ingredients in the construction of the Z_m (which will arise as the supports of suitable $(k+1)$-dimensional integral currents) is a 'thick-thin' decomposition theorem for integral currents in metric spaces admitting a polynomial isoperimetric inequality, see Theorem 4.1, where 'thick' and 'thin' are to be understood in terms of volume growth. In the special case that X admits an isoperimetric inequality of Euclidean type for $\mathbf{I}_{k-1}(X)$ and that $T \in \mathbf{I}_k(X)$ has no boundary this theorem asserts that T decomposes into the sum $T = R + T_1 + T_2 + \dots$ of integral cycles such that R is thick and all T_i are thin in the following sense: The mass of R contained in each ball $B(x, r)$ with $x \in \operatorname{spt} R$ is at least γr^k whenever $r \in [0, \delta \mathbf{M}(R)^{\frac{1}{k}}]$; furthermore each T_i has mass at most $\delta^k \gamma \mathbf{M}(T)$ and the decomposition does not add much mass, that is $\mathbf{M}(R) + \sum \mathbf{M}(T_i) \leq (1 + \lambda)\mathbf{M}(T)$. Here, $\lambda, \delta \in (0, 1)$ can be chosen arbitrarily and $\gamma \in (0, 1)$ only depends on λ. This decomposition theorem can be used to show that, if (1) fails, there exists $\varepsilon_0 > 0$ and a sequence $R_m \in \mathbf{I}_k(X)$ of cycles such that $r_m := \mathbf{M}(R_m)^{\frac{1}{k}} \to \infty$ and

$$\mathrm{Fillvol}(R_m) \geq \varepsilon_0 \mathbf{M}(R_m)^{\frac{k+1}{k}} \quad \text{for all } m \in \mathbb{N} \tag{3}$$

and with the property that the supports $\operatorname{spt} R_m$ form an equi-compact and equi-bounded sequence when endowed with the rescaled metric $\frac{1}{r_m} d_X$. A standard argument involving isoperimetric inequalities of Euclidean type, see Proposition 4.2, then shows that there exist fillings $S_m \in \mathbf{I}_{k+1}(X)$ of R_m with an isoperimetric bound on mass and such that the supports $Z_m := \operatorname{spt} S_m$ also form an equi-compact and equi-bounded sequence when endowed with the rescaled metric $\frac{1}{r_m} d_X$. In particular, a subsequence converges to some compact metric space Z in the Gromov-Hausdorff sense. In order to prove that Z receives a biLipschitz image as claimed,

one 'pushes forward' the S_m to Z, a subsequence of which converges to an integral current $S \in \mathbf{I}_{k+1}(Z)$ by the closure and compactness theorems of Ambrosio-Kirchheim. The main theorem in [29] then guarantees that $S \neq 0$ as otherwise (3) would be violated. (It is this theorem that uses the hypothesis that X admits cone type inequalities.) The support of S then contains a biLipschitz image of some compact set $K \subset \mathbb{R}^{k+1}$ with $\mathcal{L}^{k+1}(K) > 0$, thus a contradiction.
A short description of the ideas used to prove the decomposition theorem alluded to above will be given after its statement in Section 4.

1.2. **Additional results.** The following definitions generalize the notions of cone type inequalities and of isoperimetric inequalities of Euclidean type, respectively. In some sense they are of large scale flavor.

Definition 1.6. *Let $k \in \mathbb{N}$ and $\nu, \varrho > 0$. A complete metric space X is said to admit a diameter-volume inequality of type (ν, ϱ) for $\mathbf{I}_k(X)$ if there exists a $C \in (0, \infty)$ such that for every $T \in \mathbf{I}_k(X)$ with $\partial T = 0$ and bounded support*

$$\mathrm{Fillvol}(T) \leq C\,\mathrm{diam}(\mathrm{spt}\,T)\mathbf{M}(T) \tag{4}$$

if $\mathrm{diam}(\mathrm{spt}\,T) \leq 1$ *and*

$$\mathrm{Fillvol}(T) \leq C[\mathrm{diam}(\mathrm{spt}\,T)]^{\nu}\mathbf{M}(T)^{\varrho} \tag{5}$$

otherwise.

Easy examples of spaces admitting diameter-volume inequalities of type $(\nu, 1)$ are simply connected homogeneous nilpotent Lie groups of class ν and, more generally, metric spaces all of whose subsets B with $R := \mathrm{diam}\,B < \infty$ can be contracted along curves of length at most $A\max\{R, R^{\nu}\}$ and which satisfy a weak form of the fellow traveller property (similar to that of asynchronous combings in geometric group theory). For a precise statement in this direction see Section 3.

Definition 1.7. *Let $k \geq 2$ and $\alpha > 1$. A complete metric space X is said to admit an isoperimetric inequality of rank α for $\mathbf{I}_{k-1}(X)$ if there is a constant $D > 0$ such that*

$$\mathrm{FV}_k(r) \leq D I_{k,\alpha}(r) \tag{6}$$

for all $r \geq 0$, where $I_{k,\alpha}$ is the function given by

$$I_{k,\alpha}(r) := \begin{cases} r^{\frac{k}{k-1}} & 0 \leq r \leq 1 \\ r^{\frac{\alpha}{\alpha-1}} & 1 < r < \infty. \end{cases}$$

In [16, 6.32] the polynomial bound $r^{\frac{\alpha}{\alpha-1}}$ was termed an isoperimetric inequality *of rank greater than α*. Here we will use the shorter terminology *of rank α*. Isoperimetric inequalities of rank k for $\mathbf{I}_{k-1}(X)$ are exactly those of *Euclidean type*.
We then have the following theorem.

Theorem 1.8. *Let X be a complete metric space, $k \in \mathbb{N}$, $\nu, \varrho > 0$, and suppose X admits a diameter-volume inequality of type (ν, ϱ) for $\mathbf{I}_k(X)$. If $k = 1$ set $\alpha_0 := 1$. If $k \geq 2$ then suppose that X admits an isoperimetric inequality of rank α_{k-1} for*

$\mathbf{I}_{k-1}(X)$ for some $\alpha_{k-1} > 1$. If $\nu + \varrho\alpha_{k-1} > \alpha_{k-1}$ then X admits an isoperimetric inequality of rank

$$\alpha_k := 1 + \frac{\alpha_{k-1}}{\nu + \varrho\alpha_{k-1} - \alpha_{k-1}}$$

for $\mathbf{I}_k(X)$ with a constant which depends only on $k, \nu, \varrho, \alpha_{k-1}$ and the constants from the isoperimetric inequality for $\mathbf{I}_{k-1}(X)$ and the diameter-volume inequality for $\mathbf{I}_k(X)$.

The statement of the theorem can be reformulated as follows: If X admits an isoperimetric inequality for $\mathbf{I}_{k-1}(X)$ of exponent $\mu > 1$ then

$$\mathrm{FV}_{k+1}(r) \leq Dr^{\varrho+\nu-\frac{\nu}{\mu}}$$

for all $r \geq 1$ and for a suitable constant D. Theorems 3.4.C and 4.2.A in [14] and Theorem 1.2 in [28] are special cases of Theorem 1.8 with $\nu = 1$ and $\alpha_{k-1} = k$. The proof of this theorem relies on the decomposition theorem for currents but it should be noted that it does not use its full strength.
Finally, following [16] we say a complete metric space X admits an isoperimetric inequality of infinite (i.e. arbitrary large) rank for $\mathbf{I}_k(X)$ if for every $\varepsilon > 0$ there exists D_ε such that

$$\mathrm{FV}_{k+1}(r) \leq D_\varepsilon r^{1+\varepsilon}$$

for every $r \geq 1$.

Corollary 1.9. *Let X be a complete metric space, $\nu > 0$, $k, k' \in \mathbb{N}$ with $k' \leq k$, and suppose that X admits a diameter-volume inequality of type $(\nu, 1)$ for $\mathbf{I}_m(X)$ for each $m = k', \dots, k$. If X admits an isoperimetric inequality of infinite rank for $\mathbf{I}_{k'}(X)$ then X admits an isoperimetric inequality of infinite rank for $\mathbf{I}_k(X)$.*

If X is geodesic and Gromov hyperbolic then there exists a geodesic thickening X_ϱ of X which admits cone type inequalities for $\mathbf{I}_m(X_\varrho)$ for $m \geq 1$. Furthermore, X_ϱ is Gromov hyperbolic and admits a linear isoperimetric inequality for $\mathbf{I}_1(X_\varrho)$. By the corollary above, X_ϱ then admits an isoperimetric inequality of infinite rank for $\mathbf{I}_k(X_\varrho)$ for every $k \geq 1$. It is natural to ask:

Question: Let Y be geodesic and Gromov hyperbolic, $k \geq 2$, and suppose Y admits cone type inequalities for $\mathbf{I}_m(Y)$ for $m = 1, \dots, k$. Is it true that Y admits a linear isoperimetric inequality for $\mathbf{I}_k(Y)$?

An affirmative answer can be given under suitable conditions on the geometry on small scales, for example again via the embedding theorem in [4] and the Lipschitz extension results in [24]. The question seems to be open however even in the case of general (non-proper and non-cocompact) Gromov hyperbolic Hadamard spaces.

The structure of the paper is as follows: Section 2 contains comments on different notions of k-cycles in metric spaces. This section also provides the definition of asymptotic cones and contains comments on the asymptotic rank. In Section 3 we give conditions on a metric space that ensure a diameter-volume inequality. The purpose of Section 4 is to prove the key technical result of this paper, the decomposition theorem alluded to above. In Section 5 we use this decomposition theorem to prove Theorem 1.8. The proof of our main result, Theorem 1.3, is contained in Section 6. Finally, the proof of Theorem 1.4 and the stronger version with the filling radius functions is given in Section 7.

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2. P

This section provides references concerning integral currents, asymptotic cones, Hadamard spaces and Gromov hyperbolicity. It furthermore reviews the generalized filling volume and filling radius functions $\mathrm{FV}_{k+1}(X, L^\infty(X), r)$, respectively $\mathrm{FR}_{k+1}(X, L^\infty(X), r)$.

2.1. **Notions of k-chains and k-cycles in metric spaces.** All our results are stated and proved in the language of metric integral currents. These were introduced by Ambrosio and Kirchheim in [3] and provide a suitable notion of 'Lipschitz surfaces' in metric spaces. In Euclidean space they agree with integral currents as defined by Federer and Fleming in [12]. A complete reference concerning the Euclidean theory is given by [11]. Throughout this article we will use the notation, definitions and results from [3]. The definitions can also be found in Sections 2.4 and 2.5 of [31].

The following remarks are intended for the reader who wishes to translate the results from Section 1 to other (more geometric) notions of chains and cycles.

1. Let X be a Riemannian manifold. Then every compact oriented k-dimensional submanifold N of X induces an integral current $T \in \mathbf{I}_k(X)$ in a canonical way and the current induced by ∂N equals ∂T. Furthermore, $\|T\|$ is the volume measure on N, i.e. the measure induced by the volume form on N, and $\mathbf{M}(T) := \|T\|(X) = \mathrm{Vol}(N)$.

2. Let X be a metric simplicial complex with only finitely many isometry types of cells and all of whose cells are biLipschitz homeomorphic to Euclidean simplices. Then every oriented simplicial k-chain c in X of finite volume induces in a canonical way an integral current $T \in \mathbf{I}_k(X)$ and the integral $(k-1)$-current induced by ∂c equals ∂T. Moreover, $\mathbf{M}(T)$ is comparable to $\mathrm{Vol}(c)$.

3. Let X be a metric space. Then every singular Lipschitz k-chain $c = \sum_{i=1}^{n} m_i \varphi_i$, see [14], induces an integral current $T \in \mathbf{I}_k(X)$. If the Lipschitz maps φ_i are biLipschitz and have pairwise almost disjoint images then $\mathbf{M}(T)$ is comparable to $\mathrm{Vol}(c)$.

4. The reader who is mainly interested in (cocompact) Riemannian manifolds or in simplicial complexes (with finitely many isometry types of faces which are all biLipschitz homeomorphic to Euclidean simplices) may think in terms of Lipschitz chains or simplicial chains (as our results can be translated into these with the help of a deformation theorem).

Finally, we provide the precise definitions of the generalized filling volume/radius functions. For this, let X, Y be complete metric spaces and suppose X isometrically embeds into Y. Then the filling volume of $T \in \mathbf{I}_k(X)$ in Y is defined as

$$\mathrm{Fillvol}_Y(T) := \inf\{\mathbf{M}(S) : S \in \mathbf{I}_{k+1}(X), \partial S = T\}$$

where we agree on $\inf \emptyset = \infty$. In case $Y = X$ we have $\mathrm{Fillvol}_Y(T) = \mathrm{Fillvol}(T)$. Furthermore, for $r \geq 0$, we set

$$\mathrm{FV}_{k+1}(X, Y, r) := \sup\{\mathrm{Fillvol}_Y(T) : T \in \mathbf{I}_k(X), \partial T = 0, \mathbf{M}(T) \leq r\}.$$

Clearly, $\mathrm{FV}_{k+1}(X, X, r) = \mathrm{FV}_{k+1}(r)$ and furthermore

$$\mathrm{Fillvol}_Y(T) \leq \mathrm{Fillvol}(T) \quad \text{and} \quad \mathrm{FV}_{k+1}(X, Y, r) \leq \mathrm{FV}_{k+1}(r).$$

The left hand sides of both inequalities are smallest for $Y := L^\infty(X)$. Here, $L^\infty(X)$ is the Banach space of bounded functions on X with the supremum norm

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Similarly, the filling radius of $T \in \mathbf{I}_k(X)$ in Y is defined as

$$\mathrm{Fillrad}_Y(T) := \inf\{\varrho \geq 0 : \exists S \in \mathbf{I}_{k+1}(Y) \text{ with } \partial S = T, \operatorname{spt} S \subset B(\operatorname{spt} T, \varrho)\}$$

and furthermore

$$\mathrm{FR}_{k+1}(X, Y, r) := \sup\{\mathrm{Fillrad}_Y(T) : T \in \mathbf{I}_k(X), \partial T = 0, \mathbf{M}(T) \leq r\}.$$

The same obvious inequalities as for the filling volume hold for the filling radius, namely

$$\mathrm{Fillrad}_Y(T) \leq \mathrm{Fillrad}(T) \quad \text{and} \quad \mathrm{FR}_{k+1}(X, Y, r) \leq \mathrm{FR}_{k+1}(r),$$

and the left-hand sides of both inequalities are smallest for $Y := L^\infty(X)$.

2.2. **Curvature bounds, Gromov hyperbolicity, asymptotic cones.** For a general reference on Hadamard spaces, Gromov hyperbolicity and asymptotic cones we refer the reader e.g. to [5], [8], [6], [13]. As mentioned in the introduction our results in particular apply to Hadamard spaces and these probably form the prime class of examples for which the main results are of interest.

A metric space X is said to be geodesic if for every two points $x, y \in X$ there exists a map $c : [a, b] \to X$ and a $\lambda \geq 0$ satisfying $c(a) = x$, $c(b) = y$, and $d(c(t), c(t')) = \lambda|t - t'|$ for all $t, t' \in [a, b]$. Such a map is called a constant-speed geodesic joining x and y. Furthermore, X is said to be CAT(0) if it is geodesic and if every geodesic triangle is at least as thin as a comparison triangle in Euclidean space. Simply connected Riemannian manifolds of non-positive sectional curvature are examples of CAT(0)-spaces. Following [5] we call a complete CAT(0)-space a Hadamard space. A geodesic metric space is said to be Gromov hyperbolic if there exists a $\delta > 0$ such that every geodesic triangle is δ-thin, i.e. each side is contained in the δ-neighborhood of the union of the other two sides. This does not imply any restriction on the small scale and should roughly be thought of as a negative upper curvature bound in the large.

We finally give the definition of an asymptotic cone. As a general reference we mention [21]. A non-principal ultrafilter on $\mathbb{N}$ is a finitely additive probability measure ω on $\mathbb{N}$ together with the σ-algebra of all subsets such that ω takes values in $\{0, 1\}$ only and $\omega(A) = 0$ whenever $A \subset \mathbb{N}$ is finite. Using Zorn's lemma it is not difficult to establish the existence of non-principal ultrafilters on $\mathbb{N}$, see e.g. Exercise I.5.48 in [8]. It is also easy to prove the following fact. If (Y, τ) is a compact topological Hausdorff space then for every sequence $(y_m)_{m\in\mathbb{N}} \subset Y$ there exists a unique point $y \in Y$ such that

$$\omega(\{m \in \mathbb{N} : y_m \in U\}) = 1$$

for every $U \in \tau$ containing y. We will denote this point by $\lim_\omega y_m$.
Let now (X, d) be a metric space and fix a non-principal ultrafilter ω on $\mathbb{N}$, a basepoint $\star \in X$ and a sequence $r_m \nearrow \infty$. Define an equivalence relation on the set of sequences $(x_m)_{m\in\mathbb{N}} \subset X$ satisfying

$$\sup_m \frac{1}{r_m} d(\star, x_m) < \infty \tag{7}$$

by

$$(x_m) \sim (x'_m) \quad \text{if and only if} \quad \lim_\omega \frac{1}{r_m} d(x_n, x'_n) = 0.$$

Definition 2.1. *The asymptotic cone* $(X, r_m^{-1}d, \star)_\omega$ *is the set of equivalence classes of sequences* $(x_m) \subset X$ *satisfying* (7) *together with the metric given by*

$$d_\omega([(x_m)], [(x'_m)]) := \lim_\omega \frac{1}{r_m} d(x_m, x'_m).$$

The following are easy to verify: Let (X, d) be a metric space and X_ω an asymptotic cone of X. Then X_ω is complete. Furthermore, if X is geodesic then so is X_ω. If X is a Hadamard space then so is X_ω. For further properties we refer the reader to [21].

2.3. **The asymptotic rank of a metric space.** The following reformulation of Definition 1.2 is a direct consequence of the Rademacher type theorem for metric space valued Lipschitz maps in [18].

Lemma 2.2. *Let X be a metric space. Then* $\operatorname{asrk}(X)$ *is the supremum over $n \in \mathbb{N}$ for which there exists an n-dim. normed space V, subsets $S_j \subset X$ and a sequence $R_j \to \infty$ such that $\frac{1}{R_j} S_j \to B(0,1) \subset V$ in the Gromov-Hausdorff sense, where* $B(0,1)$ *denotes the closed unit ball in V.*

We obtain the following simple relations.

Proposition 2.3. *Let X be an arbitrary metric space. Then the following properties hold:*

(i) $\operatorname{asrk}(X) \leq \sup\{\operatorname{Topdim}(C) : C \subset X_\omega$ *cpt,* X_ω *an asymptotic cone of* $X\}$;
(ii) $\operatorname{asrk}(X)) \geq \sup\{n \in \mathbb{N} : \exists\psi : \mathbb{R}^n \to X$ *quasi-isometric*$\}$.

Furthermore, if X admits isoperimetric inequalities of Euclidean type for $\mathbf{I}_m(X)$ for all $m \geq 1$ then $\operatorname{asrk}(X) - 1$ *equals the supremum over all $n \in \mathbb{N}$ such that there exists an asymptotic cone of X which contains a non-trivial integral n-cycle.*

The main reason for using the terminology 'asymptotic rank' is its equivalence to the Euclidean rank in the case of proper cocompact Hadamard spaces. The following result is a direct consequence of Theorems A, C and D in [19].

Theorem 2.4. *Let X be a metric space. If X is a Hadamard space then* $\operatorname{asrk}(X)$ *is the maximal geometric dimension of an asymptotic cone of X. If X is a proper cocompact length space with a convex metric then*

$$\operatorname{asrk}(X) = \sup\{n \in \mathbb{N} : \exists V \text{ } n\text{-dim. normed space and } \psi : V \to X \text{ isometric}\}.$$

In particular, if X is a proper cocompact Hadamard space then $\operatorname{asrk}(X)$ *equals its Euclidean rank.*

For the definition of geometric dimension see [19]. Here X is said to have a convex metric if for every pair of constant-speed geodesic segments $c_1, c_2 : [0,1] \to X$ the function $t \mapsto d(c_1(t), c_2(t))$ is convex. Moreover, the Euclidean rank of a proper cocompact Hadamard space X is by definition the maximal $n \in \mathbb{N}$ such that $\mathbb{R}^n$ embeds isometrically into X.
Clearly, a geodesic Gromov hyperbolic metric space has asymptotic rank 1. The converse is not true in general. However, if X is a geodesic metric space which admits a quadratic isoperimetric inequality for curves (note that this is equivalent to admitting a cone type inequality for loops) and has $\operatorname{asrk}(X) = 1$, then X is Gromov hyperbolic.

3. M -

We first recall the definition of the product of a current with an interval, see [3] and [28].

3.1. **Products of currents.** Let X be a complete metric space and endow $[0,1]\times X$ with the Euclidean product metric. Given a Lipschitz function f on $[0,1]\times X$ and $t \in [0,1]$ we define the function $f_t : X \longrightarrow \mathbb{R}$ by $f_t(x) := f(t,x)$. With every $T \in \mathbf{N}_k(X)$, $k \geq 1$, and every $t \in [0,1]$ we associate the normal k-current on $[0,1]\times X$ given by the formula

$$([t]\times T)(f, \pi_1, \dots, \pi_k) := T(f_t, \pi_{1\,t}, \dots, \pi_{k\,t}).$$

The product of a normal current with the interval $[0,1]$ is defined as follows.

Definition 3.1. *For a normal current $T \in \mathbf{N}_k(X)$ the functional $[0,1]\times T$ on $\mathcal{D}^{k+1}([0,1]\times X)$ is given by*

$$([0,1]\times T)(f, \pi_1, \dots, \pi_{k+1}) := \sum_{i=1}^{k+1}(-1)^{i+1}\int_0^1 T\left(f_t \frac{\partial \pi_{i\,t}}{\partial t}, \pi_{1\,t}, \dots, \pi_{i-1\,t}, \pi_{i+1\,t}, \dots, \pi_{k+1\,t}\right) dt$$

for $(f, \pi_1, \dots, \pi_{k+1}) \in \mathcal{D}^{k+1}([0,1]\times X)$.

Note that $\mathcal{D}^m(Y)$ is the space of generalized differential forms, i.e. the family of tuples $(g, \tau_1, \dots, \tau_m)$ of Lipschitz functions on Y with g bounded. The proof of the following result is analogous to that of [3, Proposition 10.2 and Theorem 10.4].

Theorem 3.2. *For every $T \in \mathbf{N}_k(X)$, $k \geq 1$, with bounded support the functional $[0,1]\times T$ is a $(k+1)$-dimensional normal current on $[0,1]\times X$ with boundary*

$$\partial([0,1]\times T) = [1]\times T - [0]\times T - [0,1]\times \partial T.$$

Moreover, if $T \in \mathbf{I}_k(X)$ *then* $[0,1] \times T \in \mathbf{I}_{k+1}([0,1] \times X)$.

3.2. **Diameter-volume inequalities and generalized combings.** Let (X, d) be a metric space, $B \subset X$ with $\operatorname{diam} B < \infty$, and $h, H > 0$. Suppose there exists a Lipschitz map $\varphi : [0,1] \times B \to X$ with the following properties:

(i) There exists $x_0 \in X$ such that $\varphi(0, x) = x_0$ and $\varphi(1, x) = x$ for all $x \in B$.
(ii) The lengths of the curves $t \mapsto \varphi(t, x)$, $x \in B$, are bounded above by h.
(iii) For every $x \in B$ there exists a relatively open neighborhood $U_x \subset B$ of x and a continuous family $\varrho_{x'}$, $x' \in U$, of reparametrizations of $[0,1]$ such that $\varrho_x(t) = t$ and

$$d(\varphi(t, x), \varphi(\varrho_{x'}(t), x')) \leq H d(x, x') \quad \text{for all } x' \in U_x \text{ and all } t \in [0,1].$$

Here, a map $\nu : [0,1] \to [0,1]$ is said to be a reparametrization of $[0,1]$ if it is continuous, non-decreasing and satisfies $\nu(0) = 0$ and $\nu(1) = 1$. We call φ as above a Lipschitz contraction of B with parameters (h, H). If $h, H : [0, \infty) \to [0, \infty]$ are continuous functions such that every subset $B \subset X$ with $R := \operatorname{diam} B < \infty$ has a Lipschitz contraction with parameters $(h(R), H(R))$ then we say X admits generalized combings with length function h and distortion function H.

Examples:

(a) Let X be a Banach space or a geodesic metric space with convex metric (e.g. a Hadamard space). Then X admits generalized combings along straight lines or the unique geodesics, respectively, with length function $h(R) := R$ and distortion function $H(R) = 1$. In these examples one may choose $\varrho_{x'}(t) = t$ for all x'.
(b) If X is a simply connected homogeneous nilpotent Lie group of class c then X admits generalized combings with length function

$$h(R) := A \max\{R, R^c\}$$

and distortion function $H(R) \equiv L$ for constants A, L, see [26].

We now prove the following proposition.

Proposition 3.3. *Let X be a complete metric space, $T \in \mathbf{I}_k(X)$ a cycle with bounded support and $h, H > 0$. Suppose there exists a Lipschitz contraction φ of $B := \operatorname{spt} T$ with parameters (h, H). Then there exists $S \in \mathbf{I}_{k+1}(X)$ with $\partial S = T$ and such that*

$$\mathbf{M}(S) \leq [k(k+1)]^{\frac{k}{2}} h H^k \mathbf{M}(T).$$

Before turning to the proof we mention the following corollary, which is an immediate consequence of the proposition. For this let $\nu \geq 1$, $\mu \geq 0$ and $A, L > 0$ and define

$$h_\nu(r) := \begin{cases} Ar & 0 \leq r \leq 1 \\ Ar^\nu & 1 < r < \infty \end{cases} \quad \text{and} \quad H_\mu(r) := \begin{cases} L & 0 \leq r \leq 1 \\ Lr^\mu & 1 < r < \infty. \end{cases}$$

Corollary 3.4. *Let X be a complete metric space and $k \geq 1$. If X admits generalized combings with length function h_ν and distortion function H_μ then X admits a diameter-volume inequality of type $(\nu + k\mu, 1)$ for $\mathbf{I}_k(X)$.*

Proof of Proposition 3.3. By Theorem 4.5 of [3] it is enough to consider the case $T = \psi_\#[\![\theta]\!]$ for a biLipschitz map $\psi : K \subset \mathbb{R}^k \to X$ and $\theta \in L^1(K, \mathbb{Z})$. Set $B := \operatorname{spt} T$ and let φ be Lipschitz contraction of B. Set $S := \varphi_\#([0,1] \times T)$ and note that $S \in \mathbf{I}_{k+1}(X)$ and $\partial S = T$ by Theorem 3.2. Let f and $\pi_1, \dots, \pi_{k+1}$ be Lipschitz functions on X with f bounded and such that $\operatorname{Lip}(\pi_i) \leq 1$ for all i. We define $\tilde{\varphi}(t,z) := \varphi(t, \psi(z))$ for $t \in [0,1]$ and $z \in K$ and $\pi := (\pi_1, \dots, \pi_{k+1})$. Let $(t,z) \in [0,1]\times K$ be such that $\pi \circ \tilde{\varphi}$ is differentiable at (t,z) with non-degenerate differential, which we denote by Q, ψ is metrically differentiable at z in the sense of [18] and the curve $\gamma_z(t) := \tilde{\varphi}(t,z)$ is metrically differentiable at t. We may assume without loss of generality that $\pi \circ \tilde{\varphi}(t,z) = 0$. Denote by P the orthogonal projection of $\mathbb{R}^{k+1}$ onto the orthogonal complement of $Q(\mathbb{R} \times \{0\})$. We claim that

$$\|P(Q(0,v))\| \leq \sqrt{k+1}H \operatorname{md}\psi_z(v)$$

for all $v \in \mathbb{R}^k$. In order to see this fix $v \in \mathbb{R}^k \backslash \{0\}$ and choose for each $r > 0$ sufficiently small a t_r with $t = \varrho_{\psi(z+rv)}(t_r)$, where $\varrho_{x'}$ denotes the family of reparametrizations of $[0,1]$ around $\psi(z)$. It is easy to see that $|t_r - t| \leq Cr$ for some constant C and all $r > 0$ sufficiently small. It then follows that

$$\begin{aligned}
\|P(Q(0,v))\| &= \lim_{r\searrow 0} \frac{1}{r}\|P(\pi\circ\varphi(t,\psi(z+rv)) - \pi\circ\varphi(t_r,\psi(z))) + P(\pi\circ\varphi(t_r,\psi(z)))\| \\
&\leq \operatorname{Lip}(\pi) \limsup_{r\searrow 0} \frac{1}{r} d(\varphi(t,\psi(z+rv)), \varphi(t_r,\psi(z))) \\
&\leq \sqrt{k+1}H \operatorname{md}\psi_z(v).
\end{aligned}$$

This proves the claim and furthermore yields

$$|\det(P\circ Q)| \leq (k+1)^{\frac{k}{2}} H^k \mathbf{J}_1(\operatorname{md}(\gamma_z)_t)\mathbf{J}_k(\operatorname{md}\psi_z).$$

We use this, the area formula in [18] and Lemma 9.2 and Theorem 9.5 in [3] to conclude

$$\begin{aligned}
&|S(f,\pi_1,\dots,\pi_{k+1})| \\
&\quad\leq \int_{[0,1]\times K} \left|\theta(z) f(\tilde{\varphi}(t,z)) \det(D_{(t,z)}(\pi\circ\tilde{\varphi}))\right| d\mathcal{L}^{k+1}(t,z) \\
&\quad\leq (k+1)^{\frac{k}{2}} H^k \int_{[0,1]\times K} |\theta(z) f(\varphi(t,\psi(z)))| \mathbf{J}_1(\operatorname{md}(\gamma_z)_t)\mathbf{J}_k(\operatorname{md}\psi_z) d\mathcal{L}^{k+1}(t,z). \\
&\quad= [k(k+1)]^{\frac{k}{2}} H^k \int_{[0,1]\times X} |f(\varphi(t,x)| \mathbf{J}_1(\operatorname{md}(\gamma_{\psi^{-1}(x)})_t) d(\mathcal{L}^1 \times \|T\|)(t,x)
\end{aligned}$$

and thus

$$\|S\| \leq [k(k+1)]^{\frac{k}{2}} H^k \varphi_\#[g(t,x) d(\mathcal{L}^1 \times \|T\|)]$$

with $g(t,x) := \mathbf{J}_1(\operatorname{md}(\gamma_{\psi^{-1}(x)})_t)$. This completes the proof. □

Diameter-volume inequalities can furthermore be established for spaces with nice local geometry on which asynchronously combable groups with polynomial length functions act properly and cocompactly by isometries. See for example Chapter 10 of [10] and Theorem 5.4.1 in the notes [27].

4. A

Set $\Lambda := \{(k,\alpha) \in \mathbb{N} \times (1,\infty) : k \geq 2\} \cup \{(1,0)\}$, let $\gamma \in (0,\infty)$ and define auxiliary functions by

$$F_{1,0,\gamma}(r) = \gamma r \quad \text{and} \quad G_{1,0}(r) = r$$

and for $(k,\alpha) \in \Lambda \backslash \{(1,0)\}$

$$F_{k,\alpha,\gamma}(r) := \begin{cases} \gamma \cdot r^k & 0 \leq r \leq 1 \\ \gamma \cdot r^\alpha & 1 < r < \infty \end{cases}$$

and

$$G_{k,\alpha}(r) := \begin{cases} r^{\frac{1}{k}} & 0 \leq r \leq 1 \\ r^{\frac{1}{\alpha}} & 1 < r < \infty. \end{cases}$$

As mentioned in the introduction the following 'thick-thin' decomposition theorem, the principle result of this section, plays a crucial role in the proof of our main result.

Theorem 4.1. *Let X be a complete metric space, $(k,\alpha) \in \Lambda$, and suppose in case $k \geq 2$ that X admits an isoperimetric inequality of rank α for $\mathbf{I}_{k-1}(X)$. Then for every $\lambda \in (0,1)$ there exists a $\gamma \in (0,1)$ with the following property. Abbreviate $F := F_{k,\alpha,\gamma}$ and $G := G_{k,\alpha}$ and let $\delta \in (0,1)$. For every $T \in \mathbf{I}_k(X)$ there exist $R \in \mathbf{I}_k(X)$ and $T_j \in \mathbf{I}_k(X)$, $j \in \mathbb{N}$, such that*

$$T = R + \sum_{i=j}^{\infty} T_j$$

and for which the following properties hold:

(i) *$\partial R = \partial T$ and $\partial T_j = 0$ for all $j \in \mathbb{N}$;*

(ii) *For all $x \in \operatorname{spt} R \backslash \operatorname{spt} \partial T$ and all $0 \leq r \leq \min\{5\delta G(\mathbf{M}(R)), \operatorname{dist}(x, \operatorname{spt} \partial T)\}$*

$$\|R\|(B(x,r)) \geq \frac{1}{2} 5^{-(k+\alpha)} F(r);$$

(iii) *$\mathbf{M}(T_j) \leq (1+\lambda)\nu\gamma\mathbf{M}(T)$ for all $j \in \mathbb{N}$, where $\nu := \delta$ if $k = 1$ or $\nu := \max\{\delta^k, \delta^\alpha\}$ otherwise;*

(iv) *$\operatorname{diam}(\operatorname{spt} T_j) \leq 4G\left(\gamma^{-1} \frac{2}{1-\lambda} 5^{k+\alpha} \mathbf{M}(T_j)\right)$;*

(v) *$\mathbf{M}(R) + \frac{1-\lambda}{1+\lambda} \sum_{i=1}^{\infty} \mathbf{M}(T_i) \leq \mathbf{M}(T)$.*

For the exact value of γ see the beginning of the proof of Proposition 4.7. Loosely speaking, the theorem asserts that T can be decomposed into a thick part R, that is with the good volume growth (ii), and relatively small additional cycles T_j which are round (property (iv) above) in the sense of [14]. A simple and illustrative example is given by the integral current T built from a large k-dimensional sphere with (possibly infinitely many) very thin tentacles glued on. Roughly, the thick-thin decomposition of T is obtained by first chopping off all the tentacles and replacing them by suitable caps. This yields the thick part R of T, which turns out to be roughly the original sphere. The cycles T_j come from a suitable decomposition of the tentacles into round cycles. Such a decomposition into round cycles was exhibited in [14] and [28] to prove the isoperimetric inequality of Euclidean type. As our example suggests, a round cycle need not be thick.

The proof of Theorem 4.1 can be briefly described as follows: A point $x \in \operatorname{spt} T$ is said to belong to the thin part $\Omega(T)$ of T if (ii) fails around x on a suitable scale. An intermediate step in the proof is Proposition 4.7, which yields a decomposition $T = R + T_1 + \cdots + T_N$ which satisfies (i), (iii), (iv), (v) and

$$\sum_{i=1}^{N} \mathbf{M}(T_i) \geq \lambda' \|T\|(\Omega(T))$$

for a suitable $\lambda' > 0$. (Successive application of this in fact yields the theorem.) The proof of Proposition 4.7 is achieved by cutting away neighborhoods around points in $\Omega(T)$ and replacing them by smaller pieces. The crucial ingredient for this is the analytic Lemma 4.5 and the Vitali covering type argument contained in Lemma 4.6.

In the proof of Theorem 4.1 we will need the following result, which generalizes Lemma 3.4 of [28] and partially Lemma 3.1 of [30].

Proposition 4.2. *Let X be a complete metric space, $k \geq 2$, $\alpha > 1$, and suppose that X admits an isoperimetric inequality of rank α for $\mathbf{I}_{k-1}(X)$ with a constant $D_{k-1} \in [1, \infty)$. Then for every $T \in \mathbf{I}_{k-1}(X)$ with $\partial T = 0$ there exists an $S \in \mathbf{I}_k(X)$ with $\partial S = T$, satisfying* (6) *and with the property that for every $x \in \operatorname{spt} S$ and every $0 \leq r \leq \operatorname{dist}(x, \operatorname{spt} T)$ we have*

$$\|S\|(B(x, r)) \geq F_{k,\alpha,\mu}(r)$$

where

$$\mu := \min\left\{\frac{1}{(3D_{k-1})^{k-1}\alpha_1^k}, \frac{1}{(3D_{k-1})^{\alpha-1}\alpha_1^\alpha}\right\}$$

with $\alpha_1 := \max\{k, \alpha\}$.

The proof relies on the arguments contained in [3, Theorem 10.6].

Proof. Let $\mathcal{M}$ denote the complete metric space consisting of all $S \in \mathbf{I}_k(X)$ with $\partial S = T$ and endowed with the metric given by $d_{\mathcal{M}}(S, S') := \mathbf{M}(S - S')$. Choose an $\tilde{S} \in \mathcal{M}$ satisfying $\mathbf{M}(\tilde{S}) \leq D_{k-1} I_{k,\alpha}(\mathbf{M}(T))$. By a well-known variational principle (see e.g. [9]) there exists an $S \in \mathcal{M}$ with $\mathbf{M}(S) \leq \mathbf{M}(\tilde{S})$ and such that the function

$$S' \mapsto \mathbf{M}(S') + \frac{1}{2}\mathbf{M}(S - S')$$

has a minimum at $S' = S$. Let $x \in \operatorname{spt} S \setminus \operatorname{spt} T$ and set $R := \operatorname{dist}(x, \operatorname{spt} \partial T)$. We claim that if $r \in (0, R)$ then

$$\|S\|(B(x, r)) \geq \begin{cases} \frac{r^k}{(3D_{k-1})^{k-1}\alpha_1^k} & r \leq 3D_{k-1}\alpha_1 \\ \frac{r^\alpha}{(3D_{k-1})^{\alpha-1}\alpha_1^\alpha} & r > 3D_{k-1}\alpha_1. \end{cases} \tag{8}$$

First note that the slicing theorem [3, Theorems 5.6 and 5.7] implies that for almost every $r \in (0, R)$ the slice $\partial(S \llcorner B(x, r))$ exists, has zero boundary, and belongs to $\mathbf{I}_{k-1}(X)$. For an $S_r \in \mathbf{I}_k(X)$ with $\partial S_r = \partial(S \llcorner B(x, r))$ the integral current

$S \llcorner (X \backslash B(x,r)) + S_r$ has boundary T and thus, comparison with S yields

$$\mathbf{M}(S \llcorner (X \backslash B(x,r)) + S_r) + \frac{1}{2}\mathbf{M}(S \llcorner B(x,r) - S_r) \geq \mathbf{M}(S).$$

If, moreover, S_r is chosen such that $\mathbf{M}(S_r) \leq D_{k-1} I_{k,\alpha}(\mathbf{M}(\partial(S \llcorner B(x,r))))$ then it follows that

$$\mathbf{M}(S \llcorner B(x,r)) \leq 3\mathbf{M}(S_r) \leq 3D_{k-1} I_{k,\alpha}(\mathbf{M}(\partial(S \llcorner B(x,r))))$$

and consequently,

$$\beta(r) \leq 3D_{k-1} I_{k,\alpha}(\beta'(r)) \tag{9}$$

for almost every $r \in (0,R)$, where $\beta(r) := \|S\|(B(x,r))$.
Set $\overline{r} := \sup\{r \in [0,R] : \beta(r) \leq 3D_{k-1}\}$ and observe that for almost every $r \in (0, \overline{r})$

$$\frac{d}{dr}\left[\beta(r)^{\frac{1}{k}}\right] = \frac{\beta'(r)}{k\beta(r)^{\frac{k-1}{k}}} \geq \frac{1}{(3D_{k-1})^{\frac{k-1}{k}} k}.$$

This yields

$$\beta(r) \geq \frac{r^k}{(3D_{k-1})^{k-1} k^k}$$

for all $r \in [0, \overline{r}]$ and consequently

$$\beta(r) \geq \frac{r^k}{(3D_{k-1})^{k-1} \alpha_1^k}$$

for all $r \in [0, \overline{R}]$, where $\overline{R} := \min\{R, 3D_{k-1}\alpha_1\}$. Indeed, it is clear that $\overline{r} \leq \overline{R}$ and in case $\overline{r} < \overline{R}$ we furthermore have

$$\beta(r) \geq \beta(\overline{r}) \geq 3D_{k-1} \geq \frac{r^k}{(3D_{k-1})^{k-1} \alpha_1^k}$$

for all $r \in [\overline{r}, \overline{R}]$. This proves (8) for $r \in [0, \overline{R}]$. Now, if $R > 3D_{k-1}\alpha_1$ then for almost every $r \in [3D_{k-1}\alpha_1, R]$

$$3D_{k-1} \leq \beta(\overline{r}) \leq \beta(r) \leq 3D_{k-1} I_{k,\alpha}(\beta'(r))$$

and hence $\beta'(r) \geq 1$. It follows that

$$\frac{d}{dr}\left[\beta(r)^{\frac{1}{\alpha}}\right] = \frac{\beta'(r)}{\alpha\beta(r)^{\frac{\alpha-1}{\alpha}}} \geq \frac{1}{(3D_{k-1})^{\frac{\alpha-1}{\alpha}} \alpha_1}$$

and thus

$$\beta(r)^{\frac{1}{\alpha}} \geq \beta(3D_{k-1}\alpha_1)^{\frac{1}{\alpha}} + \frac{r - 3D_{k-1}\alpha_1}{(3D_{k-1})^{\frac{\alpha-1}{\alpha}} \alpha_1} \geq \frac{r}{(3D_{k-1})^{\frac{\alpha-1}{\alpha}} \alpha_1}.$$

This concludes the proof of (8). In order to finish the proof of the proposition it is enough to show the statement for $r \in [1, 3D_{k-1}\alpha_1]$, since the other cases are direct consequences of (8). We simply calculate

$$\|S\|(B(x,r)) \geq \frac{r^k}{(3D_{k-1})^{k-1}\alpha_1^k} \geq \frac{r^\alpha}{(3D_{k-1})^{k-1}\alpha_1^k (3D_{k-1}\alpha_1)^{\alpha-k}}$$

to obtain the desired inequality. □

A direct consequence of the proposition is the following estimate on the filling radius.

Corollary 4.3. *Let X be a complete metric space, $k \geq 2$, $\alpha > 1$, and suppose that X admits an isoperimetric inequality of rank α for $\mathbf{I}_{k-1}(X)$. Then for every $T \in \mathbf{I}_{k-1}(X)$ with $\partial T = 0$ we have*

$$\operatorname{Fillrad}_X(T) \leq G_{k,\alpha}(\mu^{-1}\operatorname{Fillvol}_X(T)) \leq \begin{cases} \mu' \mathbf{M}(T)^{\frac{1}{k-1}} & \mathbf{M}(T) \leq 1 \\ \mu' \mathbf{M}(T)^{\frac{1}{\alpha-1}} & \mathbf{M}(T) > 1, \end{cases}$$

where

$$\mu' := \max\left\{\left(\frac{D_k}{\mu}\right)^{\frac{1}{k}}, \left(\frac{D_k}{\mu}\right)^{\frac{1}{\alpha}}\right\}.$$

4.1. **An analytic lemma.** For $(k,\alpha) \in \Lambda$ and $\gamma \in (0,\infty)$ we first define an auxiliary function by

$$H_{1,0,\gamma}(r) = \gamma$$

and

$$H_{k,\alpha,\gamma}(r) := \begin{cases} \gamma^{\frac{1}{k}} \cdot r^{\frac{k-1}{k}} & 0 \leq r \leq \gamma \\ \gamma^{\frac{1}{\alpha}} \cdot r^{\frac{\alpha-1}{\alpha}} & \gamma < r < \infty \end{cases}$$

if $k \geq 2$. For the convenience of the reader we summarize some simple properties of the auxiliary functions thus far defined. Their properties will be used in the sequel without explicit mentioning.

Lemma 4.4. *Let $(k,\alpha) \in \Lambda$ and $\gamma \in (0,\infty)$ and set $F := F_{k,\alpha,\gamma}$, $G := G_{k,\alpha}$, $H := H_{k,\alpha,\gamma}$, and $I := I_{k,\alpha}$. Then the following properties hold:*

(i) *For all $r \geq 0$ we have $F(5r) \leq 5^{k+\alpha}F(r)$;*

(ii) *If $k \geq 2$ and $\nu \geq 0$ then*

$$\min\left\{\nu^{\frac{1}{k}}, \nu^{\frac{1}{\alpha}}\right\} G(r) \leq G(\nu r) \leq \max\left\{\nu^{\frac{1}{k}}, \nu^{\frac{1}{\alpha}}\right\} G(r)$$

for all $r \geq 0$;

(iii) *If $k \geq 2$ and $\nu \geq 0$ then*

$$\gamma \min\left\{\nu^k, \nu^\alpha\right\} r \leq F(\nu G(r)) \leq \gamma \max\left\{\nu^k, \nu^\alpha\right\} r$$

for all $r \geq 0$;

(iv) *If $k \geq 2$ and $\nu \geq 0$ then*

$$\min\left\{(\gamma\nu)^{\frac{1}{k}}, (\gamma\nu)^{\frac{1}{\alpha}}\right\} r \leq G(\nu F(r)) \leq \max\left\{(\gamma\nu)^{\frac{1}{k}}, (\gamma\nu)^{\frac{1}{\alpha}}\right\} r$$

for all $r \geq 0$;

(v) *If $k = 1$ then $F'(r) = H(F(r))$ for all $r \geq 0$*

(vi) *If $k \geq 2$ then $F'(r) = kH(F(r))$ when $r \in (0,1)$ and $F'(r) = \alpha H(F(r))$ when $r > 1$;*

(vii) *If $k \geq 2$ then $I(s) + I(t) \leq I(s+t)$ for all $s, t \geq 0$;*

(viii) *If* $k \geq 2$ *and* $1 \leq \mu \leq 1/\gamma$ *then*

$$I(\mu H(r)) \leq \mu \max\left\{(\mu\gamma)^{\frac{1}{k-1}}, (\mu\gamma)^{\frac{1}{\alpha-1}}\right\} \cdot r$$

for all $r \geq 0$.

The proof is by straight-forward verification and is therefore omitted. For the proof of Theorem 4.1 we need the following analytic lemma.

Lemma 4.5. *Let* $(k, \alpha) \in \Lambda$, $\gamma \in (0, 1)$, *and abbreviate* $F := F_{k,\alpha,\gamma}$ *and* $H := H_{k,\alpha,\gamma}$. *Let furthermore* $r_0 > 0$ *and suppose* $f : [0, r_0] \to [0, \infty)$ *is non-decreasing and continuous from the right with* $f(r_0) < 5^{-(k+\alpha)} F(r_0)$ *and such that*

$$r_* := \max\{r \in [0, r_0] : f(r) \geq F(r)\} > 0.$$

Then $r_* < r_0/5$ *and there is a measurable subset* $K \subset (r_*, r_0/5)$ *of strictly positive Lebesgue measure such that*

$$f(5r) < 5^{k+\alpha} f(r) \quad \text{and} \quad f'(r) < (k + \alpha)H(f(r))$$

for every $r \in K$.

This lemma will be applied with $f(r)$ the mass in a ball of radius r of an integral current.

Proof. First of all, if $r_* \geq r_0/5$ then it follows that

$$F(r_0) \leq F(5r_*) \leq 5^{k+\alpha} F(r_*) = 5^{k+\alpha} f(r_*) \leq 5^{k+\alpha} f(r_0),$$

which contradicts the hypothesis. This proves that indeed $r_* < r_0/5$. Now suppose that for almost every $r \in (r_*, r_0/5)$ we have

$$\text{either} \quad f(5r) \geq 5^{k+\alpha} f(r) \quad \text{or} \quad f'(r) \geq (k + \alpha)H(f(r)).$$

Define

$$r'_* := \inf\left\{r \in [r_*, r_0/5] : f(5r) \geq 5^{k+\alpha} f(r)\right\},$$

where we agree on $\inf \emptyset = \infty$. It then follows that $r'_* > r_*$ since otherwise

$$F(5r_*) \leq 5^{k+\alpha} F(r_*) = 5^{k+\alpha} f(r_*) \leq f(5r_*),$$

in contradiction with the definition of r_*. If $k = 1$ then set $r''_* := \min\{r'_*, r_0/5\}$ and note that $f'(r) \geq \gamma$ for almost every $r \in (r_*, r''_*)$ and thus $f(r''_*) \geq f(r_*) + \gamma(r''_* - r_*) = \gamma r''_*$, which is impossible. If, on the other hand, $k \geq 2$ then we distinguish the following two cases. Suppose first that $r_* < 1$ and set $r''_* := \min\{1, r_0/5, r'_*\}$; observe that $r''_* > r_*$ and $f(r''_*) < \gamma$. Consequently, we have

$$\frac{d}{dr}\left[f(r)^{\frac{1}{k}}\right] = \frac{f'(r)}{k f(r)^{\frac{k-1}{k}}} \geq \frac{(k + \alpha)H(f(r))}{k f(r)^{\frac{k-1}{k}}} > \gamma^{\frac{1}{k}}$$

for almost every $r \in (r_*, r''_*)$ and hence

$$f(r''_*)^{\frac{1}{k}} > f(r_*)^{\frac{1}{k}} + \gamma^{\frac{1}{k}}(r''_* - r_*) = \gamma^{\frac{1}{k}} r''_*,$$

which is not possible. Suppose next that $r_* \geq 1$ and set $r''_* := \min\{r_0/5, r'_*\}$; observe that $r''_* > r_*$ and $f(r''_*) > \gamma$, from which we conclude analogously as above that

$$\frac{d}{dr}\left[f(r)^{\frac{1}{\alpha}}\right] = \frac{f'(r)}{\alpha f(r)^{\frac{\alpha-1}{\alpha}}} \geq \frac{(k+\alpha)H(f(r))}{\alpha f(r)^{\frac{\alpha-1}{\alpha}}} > \gamma^{\frac{1}{\alpha}}$$

for almost every $r \in (r_*, r''_*)$ and thus

$$f(r''_*)^{\frac{1}{\alpha}} > f(r_*)^{\frac{1}{\alpha}} + \gamma^{\frac{1}{\alpha}}(r''_* - r_*) > \gamma^{\frac{1}{\alpha}} r''_*,$$

again a contradiction with the definition of r_*. This concludes the proof of the lemma. ◻

4.2. **Controlling the thin parts of a current.** Let X be a complete metric space and fix $(k, \alpha) \in \Lambda$. The following set which we associate with an element $T \in \mathbf{I}_k(X)$ and constants $\gamma \in (0, 1)$ and $L \in (0, \infty]$ will sometimes be referred to as the thin part of T,

$$\Omega(T, \gamma, L) := \Big\{x \in \operatorname{spt} T : \Theta_{*k}(\|T\|, x) > \frac{\gamma}{\omega_k} \text{ and } \|T\|(B(x, r)) < \frac{1}{2}5^{-(k+\alpha)}F_{k,\alpha,\gamma}(r) \\ \text{for an } r \in [0, \min\{L, \operatorname{dist}(x, \operatorname{spt}\partial T)\}]\Big\}.$$

Note that we explicitly allow the value $L = \infty$. Furthermore, we agree on the convention $\operatorname{dist}(x, \emptyset) = \infty$. It should be remarked that $\Omega(T, \gamma, L)$ also depends on α even though we omit α in our notation. The inequality involving the lower density is satisfied for $\|T\|$-almost every $x \in \operatorname{spt} T$ if $\gamma < \omega_k k^{-k/2}$ by [3]. It is not difficult to see that $\Omega(T, \gamma, L)$ is then $\|T\|$-measurable and that, in case $\partial T = 0$, we have $\Omega(T, \gamma, \infty) = \operatorname{spt} T$ up to a set of $\|T\|$-measure zero.

Lemma 4.6. *Let X be a complete metric space, $(k, \alpha) \in \Lambda$, and $\gamma \in (0, a)$, where $a := \min\{1, \omega_k k^{-k/2}\}$. Abbreviate $F := F_{k,\alpha,\gamma}$, $G := G_{k,\alpha}$ and $H := H_{k,\alpha,\gamma}$. Let furthermore $T \in \mathbf{I}_k(X)$ and $L \in (0, \infty]$. Then there exist finitely many points $x_1, \dots, x_N \in \Omega(T, \gamma, L)$ and $s_1, \dots, s_N \in (0, \infty)$ with the following properties:*

(i) *With $A := G(\gamma^{-1}\|T\|(B(x_i, s_i)))$ we have*

$$A < s_i < \min\left\{\frac{L}{5}, \frac{1}{5}\operatorname{dist}(x_i, \operatorname{spt}\partial T), 2 \cdot 5^{k+\alpha}A\right\}$$

(ii) $B(x_i, 2s_i) \cap B(x_j, 2s_j) = \emptyset$ *for all* $i \neq j$
(iii) $T \llcorner B(x_i, s_i) \in \mathbf{I}_k(X)$
(iv) $\frac{1}{2}5^{-(k+\alpha)}F(s_i) \leq \|T\|(B(x_i, s_i)) \leq F(s_i)$
(v) $\mathbf{M}(\partial(T \llcorner B(x_i, s_i))) \leq (k+\alpha)H(\|T\|(B(x_i, s_i)))$
(vi) $\sum_{i=1}^N \|T\|(B(x_i, s_i)) \geq 5^{-(k+\alpha)}\|T\|(\Omega(T, \gamma, L))$.

We note that in the above we allow $N = 0$ if $\|T\|(\Omega(T, \gamma, L)) = 0$.

Proof. For each $x \in \Omega(T, \gamma, L)$ set

$$f_x(r) := \|T\|(B(x, r)) \qquad \text{for } r \in [0, \infty)$$

and note that f_x is non-decreasing and continuous from the right. Define furthermore

$$r_0(x) := \inf\left\{r \in [0, \min\{L, \operatorname{dist}(x, \operatorname{spt}\partial T)\}] : \|T\|(B(x,r)) < \frac{1}{2}5^{-(k+\alpha)}F(r)\right\}.$$

Since

$$\liminf_{r\searrow 0}\frac{f_x(r)}{r^k} = \omega_k\Theta_{*k}(\|T\|, x) > \gamma,$$

it follows that $r_0(x) > 0$ and

$$r_*(x) := \max\{r \in [0, r_0(x)] : f_x(r) \geq F(r)\} > 0.$$

Note that we also have

$$r_0(x) = G(\gamma^{-1}F(r_0(x))) = G(2\gamma^{-1}5^{k+\alpha}\mathbf{M}(T)) \tag{10}$$

and $f_x(r_0(x)) < 5^{-(k+\alpha)}F(r_0(x))$. Lemma 4.5 and the slicing theorem for rectifiable currents imply that there exists for each $x \in \Omega(T,\gamma,L)$ an $r(x) \in (r_*(x), r_0(x)/5)$ such that

(a) $T \llcorner B(x, r(x)) \in \mathbf{I}_k(X)$
(b) $\|T\|(B(x, r(x))) < F(r(x))$
(c) $\|T\|(B(x, 5r(x))) < 5^{k+\alpha}\|T\|(B(x, r(x)))$
(d) $\mathbf{M}(\partial(T \llcorner B(x, r(x)))) \leq f'_x(r(x)) < (k+\alpha)H(\|T\|(B(x, r(x))))$.

The points $x_1, \dots, x_N$ and the radii $s_1, \dots, s_N$ are now constructed as follows: Set $\Omega_1 := \Omega(T,\gamma,L)$ and $s_1^* := \sup\{r(x) : x \in \Omega_1\}$. From (10) it follows that $s_1^* < \infty$. Choose $x_1 \in \Omega_1$ in such a way that $r(x_1) > \frac{2}{3}s_1^*$. If $x_1, \dots, x_j$ are chosen define

$$\Omega_{j+1} := \Omega(T,\gamma,L)\backslash\bigcup_{i=1}^{j} B(x_i, 5r(x_i))$$

and

$$s_{j+1}^* := \sup\{r(x) : x \in \Omega_{j+1}\}.$$

If $\|T\|(\Omega_{j+1}) > 0$ we can choose $x_{j+1} \in \Omega_{j+1}$ such that $r(x_{j+1}) > \frac{2}{3}s_{j+1}^*$. This procedure yields (possibly finite) sequences $x_j \in \Omega_j$, $s_1^* \geq s_2^* \geq \dots \geq 0$, and $s_i := r(x_i)$. We show that for a suitably large N the so defined points and numbers have the desired properties stated in the lemma. We first note that, by (b) and the definition of $r_0(x)$,

$$\frac{1}{2}5^{-(k+\alpha)}F(s_i) \leq \|T\|(B(x_i, s_i)) < F(s_i),$$

which proves (iv). Property (i) follows from this and the fact that $s_i = G(\gamma^{-1}F(s_i))$. Furthermore, we have

$$d(x_i, x_{i+\ell}) > 5s_i = 2s_i + 3s_i > 2s_i + 2s_i^* \geq 2s_i + 2s_{i+\ell}$$

and thus we obtain (ii). Properties (iii) and (v) are direct consequences of (a) and (d), respectively. We are therefore left to show that (vi) holds for some $N \in \mathbb{N}$. On the one hand, if $\|T\|(\Omega_{n+1}) = 0$ for some $n \in \mathbb{N}$ then (c) yields

$$\sum_{i=1}^{n} \|T\|(B(x_i, s_i)) > 5^{-(k+\alpha)} \sum_{i=1}^{n} \|T\|(B(x_i, 5s_i)) \geq 5^{-(k+\alpha)} \|T\|(\Omega(T, \gamma, L)),$$

which establishes (vi) and thus the lemma with $N = n$. On the other hand, if $\|T\|(\Omega_n) > 0$ for all $n \in \mathbb{N}$ then it follows easily that $s_n^* \searrow 0$. Indeed, this is a consequence of the fact that

$$\frac{1}{2} 5^{-(k+\alpha)} \sum_{i=1}^{\infty} F\left(\frac{2}{3} s_i^*\right) < \sum_{i=1}^{\infty} \frac{1}{2} 5^{-(k+\alpha)} F(s_i) \leq \sum_{i=1}^{\infty} \|T\|(B(x_i, s_i)) \leq \mathbf{M}(T) < \infty.$$

Furthermore we claim that

$$\|T\|\left(\Omega(T, \gamma, L) \backslash \bigcup_{i=1}^{\infty} B(x_i, 5s_i)\right) = 0.$$

If this were not true we would have $x \in \Omega(T, \gamma, L) \backslash \cup_{i=1}^{\infty} B(x_i, 5s_i)$ and since $r(x) > 0$ we would obtain a contradiction with $s_i^* \searrow 0$. The rest now follows as in the case above. □

4.3. **Proof of Theorem 4.1.** The following proposition is an intermediate step on the way to the main theorem of this section. The proposition shows how to construct a suitable decomposition of a current $T \in \mathbf{I}_k(X)$ in a way that helps to reduce the set $\Omega(T, \gamma, L)$.

Proposition 4.7. *Let X be a complete metric, $(k, \alpha) \in \Lambda$ and suppose in case $k \geq 2$ that X admits an isoperimetric inequality of rank α for $\mathbf{I}_{k-1}(X)$. For every $\lambda \in (0, 1)$ there exists a $\gamma \in (0, 1)$ with the following property. Set $F := F_{k,\alpha,\gamma}$ and $G := G_{k,\alpha}$. Then for every $L \in (0, \infty]$ and $T \in \mathbf{I}_k(X)$ there is a decomposition*

$$T = R + T_1 + \cdots + T_N$$

with $R, T_i \in \mathbf{I}_k(X)$ such that

(i) $\partial R = \partial T$ *and* $\partial T_i = 0$
(ii) $\mathbf{M}(T_i) \leq (1 + \lambda) F(L/5)$
(iii) $\operatorname{diam}(\operatorname{spt} T_i) \leq 4G\left(\gamma^{-1} \frac{2}{1-\lambda} 5^{k+\alpha} \mathbf{M}(T_i)\right)$
(iv) $\mathbf{M}(R) + \frac{1-\lambda}{1+\lambda} \sum_{i=1}^{N} \mathbf{M}(T_i) \leq \mathbf{M}(T)$
(v) $\sum_{i=1}^{N} \mathbf{M}(T_i) \geq (1 - \lambda) 5^{-(k+\alpha)} \|T\|(\Omega(T, \gamma, L))$.

It should be noted that the main purpose of decreasing λ is to make (iv) more optimal.

Proof. If $k = 1$ then set $\gamma := 1/2$. If $k \geq 2$ then define

$$\gamma := \frac{1}{k + \alpha} \min\left\{A^{k-1}, A^{\alpha-1}, \omega_k k^{-\frac{k}{2}}\right\} \quad \text{with} \quad A := \frac{\lambda}{3D_{k-1}(k + \alpha)}$$

and where D_{k-1} denotes the constant in the isoperimetric inequality for $\mathbf{I}_{k-1}(X)$. We may of course assume that $D_{k-1} \geq 1$. We may furthermore assume that $\|T\|(\Omega(T, \gamma, L)) > 0$ since otherwise we can set $R := T$ and there is then nothing to prove. Let $x_1, \ldots, x_N \in \Omega(T, \gamma, L)$ and $s_1, \ldots, s_N \in (0, \infty)$ be as in Lemma 4.6. Fix

$i \in \{1, \dots, N\}$. If $k = 1$ then set $T_i := T \llcorner B(x_i, s_i)$ and note that $\mathbf{M}(\partial T_i) \leq \gamma < 1$ and thus $\partial T_i = 0$. If, on the other hand, $k \geq 2$ then choose $S_i \in \mathbf{I}_k(X)$ such that $\partial S_i = \partial(T \llcorner B(x_i, s_i))$ and with the properties of Proposition 4.2. It follows that

$$\mathbf{M}(S_i) \leq D_{k-1} I_{k,\alpha}(\mathbf{M}(\partial(T \llcorner B(x_i, s_i)))) \leq \lambda \|T\|(B(x_i, s_i)), \tag{11}$$

where for the second inequality we use the definition of γ. Next we have that $\operatorname{spt} S_i \subset B(x_i, 2s_i)$. This is indeed a consequence of Proposition 4.2 and the fact that

$$\mathbf{M}(S_i) \leq \lambda \|T\|(B(x_i, s_i)) \leq \lambda F(s_i) = \begin{cases} \lambda\gamma \cdot s_i^k & s_i \leq 1 \\ \lambda\gamma \cdot s_i^\alpha & s_i > 1 \end{cases}$$

and the choice of γ. Thus $T_i := T \llcorner B(x_i, s_i) - S_i$ satisfies $T_i \in \mathbf{I}_k(X)$, $\partial T_i = 0$ and $\operatorname{spt} T_i \subset B(x_i, 2s_i)$. From (11) we see that

$$(1 - \lambda)\|T\|(B(x_i, s_i)) \leq \mathbf{M}(T_i) \leq (1 + \lambda)\|T\|(B(x_i, s_i)) \tag{12}$$

and thus

$$\mathbf{M}(T_i) \leq (1 + \lambda)\|T\|(B(x_i, s_i)) \leq (1 + \lambda)F(s_i) \leq (1 + \lambda)F(L/5),$$

which proves (ii) of the present proposition. Note that the same conclusion holds in the case $k = 1$. We proceed as above for every $i \in \{1, \dots, N\}$ and note that in each step of the construction only the ball $B(x_i, 2s_i)$, which is disjoint from the other balls, is affected. We thus obtain cycles $T_1, \dots, T_N$ and we claim that these together with $R := T - T_1 - \cdots - T_N$ have the properties stated in the proposition. Indeed, (i) is obvious and (ii) has already been proved. As for (iii) it is enough to note that $\operatorname{diam}(\operatorname{spt} T_i) \leq 4s_i$ and

$$s_i = G(\gamma^{-1} F(s_i)) \leq G\left(\gamma^{-1} \frac{2}{1 - \lambda} 5^{k+\alpha} \mathbf{M}(T_i)\right).$$

Furthermore, by construction,

$$\begin{aligned}
\mathbf{M}(R) &\leq \|T\|\left(X \setminus \bigcup_{i=1}^N B(x_i, s_i)\right) + \lambda \sum_{i=1}^N \|T\|(B(x_i, s_i)) \\
&= \mathbf{M}(T) - (1 - \lambda) \sum_{i=1}^N \|T\|(B(x_i, s_i)) \\
&\leq \mathbf{M}(T) - \frac{1 - \lambda}{1 + \lambda} \sum_{i=1}^N \mathbf{M}(T_i)
\end{aligned}$$

from which (iv) follows. Finally, we use (vi) of Lemma 4.6 together with (12) to calculate

$$\sum_{i=1}^N \mathbf{M}(T_i) \geq (1 - \lambda) \sum_{i=1}^N \|T\|(B(x_i, s_i)) \geq (1 - \lambda)5^{-(k-\alpha)} \|T\|(\Omega(T, \gamma, L)).$$

This establishes (v) and concludes the proof of the proposition. □

We are now ready for the proof of the 'thick-thin' decomposition theorem.

Proof of Theorem 4.1. Set $R_0 := T$ and $N_0 := 0$. Successive application of Proposition 4.7 yields possibly finite sequences $(R_i), (T_j) \subset \mathbf{I}_k(X)$ and a strictly increasing sequence of integers $N_1 < N_2 < \dots$ such that for every $i \in \mathbb{N} \cup \{0\}$

$$R_i = R_{i+1} + T_{N_i+1} + \cdots + T_{N_{i+1}}$$

and such that the following properties hold:

(a) $\partial R_i = \partial T$ and $\partial T_j = 0$ for all i, j
(b) $\mathbf{M}(T_j) \leq (1+\lambda)\nu\gamma\mathbf{M}(R_i)$ for all $j \in \{N_i + 1, \dots, N_{i+1}\}$
(c) $\operatorname{diam}(\operatorname{spt} T_j) \leq 4G\left(\frac{2}{\gamma(1-\lambda)} 5^{k+\alpha}\mathbf{M}(T_j)\right)$
(d) $\mathbf{M}(R_{i+1}) + \frac{1-\lambda}{1+\lambda}\sum_{j=N_i+1}^{N_{i+1}} \mathbf{M}(T_j) \leq \mathbf{M}(R_i)$
(e) $\sum_{j=N_i+1}^{N_{i+1}} \mathbf{M}(T_j) \geq (1-\lambda)5^{-(k+\alpha)}\|R_i\|(\Omega(R_i, \gamma, L_i))$.

Here, L_i is defined by $L_i := 5\delta G(\mathbf{M}(R_i))$ and $\nu := \delta$ if $k = 1$ and $\nu = \max\{\delta^k, \delta^\alpha\}$ otherwise. We thus obtain for each $i \in \mathbb{N} \cup \{0\}$ a decomposition

$$T = R_i + \sum_{j=1}^{N_i} T_j$$

which, by property (d), satisfies

$$\mathbf{M}(R_i) + \frac{1-\lambda}{1+\lambda}\sum_{j=1}^{N_i} \mathbf{M}(T_j) \leq \mathbf{M}(T). \tag{13}$$

In particular, we have

$$\mathbf{M}(R_{i+m} - R_i) = \mathbf{M}(T_{N_i+1} + \cdots + T_{N_{i+m}}) \leq \sum_{j=N_i+1}^{\infty} \mathbf{M}(T_j) \to 0$$

as $i \to \infty$ and it thus follows that the sequence (R_i) is Cauchy with respect to the mass norm. Since the additive group of integer rectifiable k-currents together with the mass norm is complete, there exists $R \in \mathbf{I}_k(X)$ such that $\mathbf{M}(R - R_i) \to 0$ and, in particular,

$$T = R + \sum_{j=1}^{\infty} T_j.$$

Clearly, we have $\partial R = \partial T$ and thus property (i) holds. Properties (iii), (iv) and (v) are direct consequences of (b), (c) and (13), respectively. We are therefore left to establish (ii). For this let $x \in \operatorname{spt} R \backslash \operatorname{spt} \partial T$ and

$$0 < r < \min\{5\delta G(\mathbf{M}(R)), \operatorname{dist}(x, \operatorname{spt} \partial T)\}.$$

Observe that

$$\|R_i\|(B(x,t)) \to \|R\|(B(x,t))$$

and $\|R\|(B(x,t)) > 0$ for all $t \in (0, r)$. Fix $0 < s < r$ and $\varepsilon > 0$ arbitrary. By (e) and (13) we have

$$\|R_i\|(\Omega(R_i, \gamma, L_i)) \to 0$$

and thus there exists $i_0 \in \mathbb{N}$ and $x' \in \operatorname{spt} R_{i_0}$ with $d(x, x') \leq s$ and

$$\|R\|(B(x,r)) \geq (1-\varepsilon)\|R_{i_0}\|(B(x,r))$$

and such that

$$\|R_{i_0}\|(B(x', r-s)) \geq \frac{1}{2}5^{-(k+\alpha)}F(r-s).$$

It finally follows that

$$\begin{aligned}\|R\|(B(x,r)) &\geq (1-\varepsilon)\|R_{i_0}\|(B(x,r)) \\ &\geq (1-\varepsilon)\|R_{i_0}\|(B(x', r-s)) \\ &\geq \frac{1}{2}(1-\varepsilon)5^{-(k+\alpha)}F(r-s).\end{aligned}$$

Since s and ε were arbitrary this establishes (ii) and completes the proof of the theorem. □

5. P T 1.8

We start with the following simple lemma which will be needed in the sequel.

Lemma 5.1. *Let X be a complete metric space, $(k,\alpha) \in \Lambda$, and $\varepsilon, \delta > 0$. Set $F := F_{k,\alpha,\varepsilon}$ and $G := G_{k,\alpha}$ and let $R \in \mathbf{I}_k(X)$ satisfy $\partial R = 0$ and*

$$\|R\|(B(x,r)) \geq F(r)$$

for all $x \in \operatorname{spt} R$ and all $r \in [0, \delta G(\mathbf{M}(R))]$. Then there exist constants $m \in \mathbb{N}$ and $E > 0$ depending only on k, α, δ, ε and a decomposition $R = R_1 + \cdots + R_m$ with $R_i \in \mathbf{I}_k(X)$, $\partial R_i = 0$, and

(i) $\|R_i\|(B(x,r)) \geq F(r)$ *for all $x \in \operatorname{spt} R_i$ and all $r \in [0, \delta G(\mathbf{M}(R))]$*
(ii) $\mathbf{M}(R) = \mathbf{M}(R_1) + \cdots + \mathbf{M}(R_m)$
(iii) $\operatorname{diam}(\operatorname{spt} R_i) \leq EG(\mathbf{M}(R_i))$.

Proof. Set $\alpha' := \alpha$ if $k \geq 2$ or $\alpha' := 1$ if $k = 1$. Fix $x \in \operatorname{spt} R$ arbitrary and observe that

$$\|R\|\left(B\left(x, t + 2^{-1}\delta G(\mathbf{M}(R))\right) \setminus B\left(x, t - 2^{-1}\delta G(\mathbf{M}(R))\right)\right) = 0$$

for some $t \in \left[\frac{5}{2}\delta G(\mathbf{M}(R)), \frac{3}{\varepsilon}\max\{\delta^{1-k}, \delta^{1-\alpha'}\}G(\mathbf{M}(R))\right]$. Thus $R_1 := R \llcorner B(x,t)$ satisfies $R_1 \in \mathbf{I}_k(X)$, $\partial R_1 = 0$ and

$$\|R_1\|(B(x', r)) \geq F(r) \tag{14}$$

for all $x' \in \operatorname{spt} R_1$ and all $0 \leq r \leq \delta G(\mathbf{M}(R))$. In particular, we have

$$\mathbf{M}(R_1) \geq \varepsilon \min\{\delta^k, \delta^{\alpha'}\}\mathbf{M}(R) \tag{15}$$

and thus

$$\operatorname{diam}(\operatorname{spt} R_1) \leq EG(\mathbf{M}(R_1))$$

for a constant E depending only on $k, \alpha, \delta, \varepsilon$. Proceeding in the same way with $R - R_1$ one eventually obtains a decomposition $R = R_1 + \cdots + R_m$ with the desired properties. The bound on m clearly follows from (15). □

We are now ready for the proof of Theorem 1.8.

Proof of Theorem 1.8. Set $\delta := \lambda := 1/5$ and set furthermore $\alpha := 0$ in case $k = 1$ and $\alpha := \alpha_{k-1}$ otherwise. Abbreviate $F := F_{k,\alpha,\gamma}$ and $G := G_{k,\alpha}$, where γ is the constant of Theorem 4.1. Let $T \in \mathbf{I}_k(X)$ with $\partial T = 0$ and let a R, T_j be given as in Theorem 4.1. Throughout this proof, the numbers (i) through (v) will refer to the properties listed in Theorem 4.1. Furthermore all the constants E_l used will depend only on k, α, γ. Set $T_0 := R$. After possible application of Lemma 5.1 we may assume that

$$\operatorname{diam}(\operatorname{spt} T_j) \le E_1 G(\mathbf{M}(T_j))$$

for all $j \ge 0$ and for a constant E_1. Suppose first that $\mathbf{M}(T) \le 1$. Then, by (v), we have $\mathbf{M}(T_j) \le 3/2$ for all $j \ge 0$, so that

$$\operatorname{diam}(\operatorname{spt} T_j) \le E_2 \mathbf{M}(T_j)^{\frac{1}{k}}$$

for some constant E_2. Thus the diameter-volume inequality yields for each $j \ge 0$ an $S_j \in \mathbf{I}_{k+1}(X)$ with $\partial S_j = T_j$ and

$$\mathbf{M}(S_j) \le E_3 \mathbf{M}(T_j)^{\frac{k+1}{k}}$$

for some E_3 depending only on k, α, γ and C_k, ν, ϱ. This is clear if $\operatorname{diam}(\operatorname{spt} T_j) \le 1$. If, on the other hand, $\operatorname{diam}(\operatorname{spt} T_j) > 1$, then we have $\mathbf{M}(T_j) > E_2^{-k}$ and from this the inequality readily follows. Finally, we have

$$\sum_{j=0}^{\infty} \mathbf{M}(S_j) \le E_3 \sum_{j=0}^{\infty} \mathbf{M}(T_j)^{\frac{k+1}{k}} \le E_3 \left[\sum_{j=0}^{\infty} \mathbf{M}(T_j)\right]^{\frac{k+1}{k}} \le E_3 \left[\frac{3}{2}\right]^{\frac{k+1}{k}} \mathbf{M}(T)^{\frac{k+1}{k}}.$$

Therefore, $\sum_{j=0}^{n} S_j$ is a Cauchy sequence with respect to mass and therefore converges to some $S \in \mathbf{I}_{k+1}(X)$, which clearly satisfies $\partial S = T$ and

$$\mathbf{M}(S) \le \sum_{j=0}^{\infty} \mathbf{M}(S_j) \le E_3 \left[\frac{3}{2}\right]^{\frac{k+1}{k}} \mathbf{M}(T)^{\frac{k+1}{k}}.$$

This proves the theorem if $\mathbf{M}(T) \le 1$. In case $\mathbf{M}(T) > 1$ define $J := \{j \ge 0 : \operatorname{diam}(\operatorname{spt} T_j) > 1\}$. If $j \in J$ then, by the diameter-volume inequality, there exists an $S_j \in \mathbf{I}_{k+1}(X)$ with $\partial S_j = T_j$ and

$$\mathbf{M}(S_j) \le C_k \mathbf{M}(T_j)^{\frac{\nu}{\alpha} + \varrho}.$$

If, on the other hand, $j \notin J$ then, again by the diameter-volume inequality, there exists an $S_j \in \mathbf{I}_{k+1}(X)$ with $\partial S_j = T_j$ and

$$\mathbf{M}(S_j) \le C_k \operatorname{diam}(\operatorname{spt} T_j) \mathbf{M}(T_j) \le C_k \mathbf{M}(T_j).$$

Since $\nu + \varrho\alpha > \alpha$ and $\mathbf{M}(T) > 1$ we have

$$\begin{aligned}\sum_{j=0}^{\infty} \mathbf{M}(S_j) &\leq C_k \sum_{j\in J} \mathbf{M}(T_j)^{\frac{\nu}{\alpha}+\varrho} + C_k \sum_{j\notin J} \mathbf{M}(T_j)\\ &\leq C_k \left[\sum_{j\in J} \mathbf{M}(T_j)\right]^{\frac{\nu}{\alpha}+\varrho} + \frac{3}{2} C_k \mathbf{M}(T)\\ &\leq 2C_k \left[\frac{3}{2}\right]^{\frac{\nu}{\alpha}+\varrho} \mathbf{M}(T)^{\frac{\nu}{\alpha}+\varrho}.\end{aligned}$$

It now follows exactly as above that $\sum_{j=0}^{n} S_j$ converges in mass to some $S \in \mathbf{I}_{k+1}(X)$ which has the desired properties. This concludes the proof. □

6. S -E

In this section we prove the main result of this paper, Theorem 1.3. As outlined in the introduction one of the crucial ingredients in the proof is Theorem 4.1. We begin with the following simple lemma.

Lemma 6.1. *Let $k \geq 2$, $\alpha > 1$ and $\lambda, \delta \in (0,1)$. If $L > 0$ and $t_i \in (0, \delta L)$ are such that*

$$\lambda \sum_{i=1}^{\infty} t_i \leq L$$

then

$$\sum_{i=1}^{\infty} I_{k,\alpha}(t_i) \leq \frac{2(1+\delta\lambda)}{\lambda} \max\left\{(2\delta)^{\frac{1}{k-1}}, (2\delta)^{\frac{1}{\alpha-1}}\right\} I_{k,\alpha}(L).$$

Proof. Pick finitely many integer numbers $0 =: m_0 < m_1 < m_2 < \cdots < m_{j_0}$ with the property that

$$\delta L < t_{m_{i-1}+1} + \cdots + t_{m_i} < 2\delta L$$

for each $i = 1, \dots, j_0$ and

$$\sum_{n=m_{j_0}+1}^{\infty} t_n \leq \delta L.$$

Then $j_0 \leq \frac{1}{\lambda\delta}$ and hence

$$\begin{aligned}\sum_{i=1}^{\infty} I_{k,\alpha}(t_i) &\leq \sum_{i=1}^{j_0} I_{k,\alpha}(t_{m_{i-1}+1} + \cdots + t_{m_i}) + I_{k,\alpha}\left(\sum_{n=m_{j_0}+1}^{\infty} t_n\right)\\ &\leq \frac{1}{\lambda\delta} I_{k,\alpha}(2\delta L) + I_{k,\alpha}(\delta L)\\ &\leq \frac{2(1+\delta\lambda)}{\lambda} \max\left\{(2\delta)^{\frac{1}{k-1}}, (2\delta)^{\frac{1}{\alpha-1}}\right\} I_{k,\alpha}(L).\end{aligned}$$

□

Lemma 6.2. *Let X be a complete metric space, $k \geq 1$, $\alpha > 1$, and suppose X admits an isoperimetric inequality of rank α for $I_k(X)$. In case $k \geq 2$ suppose furthermore that X also admits an isoperimetric inequality of rank k for $\mathbf{I}_{k-1}(X)$. Let $\varepsilon > 0$ and $T \in \mathbf{I}_k(X)$ with $\partial T = 0$. If $\operatorname{Fillvol}(T) \geq \varepsilon I_{k+1,\alpha}(\mathbf{M}(T))$ then there exists a $T' \in \mathbf{I}_k(X)$ with $\partial T' = 0$ and the following properties:*

(i) $\operatorname{Fillvol}(T') \geq \frac{\varepsilon}{2} I_{k+1,\alpha}(\mathbf{M}(T'))$
(ii) $\mathbf{M}(T') \geq A\mathbf{M}(T)$
(iii) $\operatorname{diam}(\operatorname{spt} T') \leq B\mathbf{M}(T')^{\frac{1}{k}}$
(iv) $\|T'\|(B(x,r)) \geq Cr^k$ *for all* $r \in [0, 5\delta \mathbf{M}(T')^{\frac{1}{k}}]$.

Here, $A, B, C, \delta > 0$ are constants depending only on k, α, ε and the constants of the isoperimetric inequalities.

Proof. Set $\lambda := 1/3$ and

$$\delta := \min\left\{\frac{3}{8}, \frac{\varepsilon}{64D_k}, \left(\frac{\varepsilon}{64D_k}\right)^{\frac{\alpha-1}{k}}\right\},$$

where D_k is the isoperimetric constant for $\mathbf{I}_k(X)$. Let $T = R + \sum_{j=1}^{\infty} T_j$ be a decomposition as in Theorem 4.1. It then follows from Lemma 6.1 that

$$\begin{aligned}\operatorname{Fillvol}(T) &\leq \operatorname{Fillvol}(R) + D_k \sum_{i=1}^{\infty} I_{k+1,\alpha}(\mathbf{M}(T_j))\\ &\leq \operatorname{Fillvol}(R) + \frac{\varepsilon}{2} I_{k+1,\alpha}(\mathbf{M}(T))\end{aligned}$$

and thus

$$\operatorname{Fillvol}(R) \geq \frac{\varepsilon}{2} I_{k+1,\alpha}(\mathbf{M}(T)) \geq \frac{\varepsilon}{2} I_{k+1,\alpha}(\mathbf{M}(R)). \tag{16}$$

This together with the isoperimetric inequality for $\mathbf{I}_k(X)$ yields

$$\mathbf{M}(R) \geq \min\left\{\left(\frac{\varepsilon}{2D_k}\right)^{\frac{k}{k+1}}, \left(\frac{\varepsilon}{2D_k}\right)^{\frac{\alpha-1}{\alpha}}\right\}\mathbf{M}(T).$$

Let $R = R_1 + \cdots + R_m$ be a decomposition of R as in Lemma 5.1. By (16) and the special properties of the decomposition there exists an i such that $T' := R_i$ satisfies

$$\operatorname{Fillvol}(T') \geq \frac{\varepsilon}{2} I_{k+1,\alpha}(\mathbf{M}(T')).$$

By Lemma 5.1, T' satisfies all the desired properties. □

We are now ready for the proof of the sub-Euclidean isoperimetric inequality.

Proof of Theorem 1.3. We argue by contradiction and suppose therefore that

$$\limsup_{r\to\infty} \frac{\operatorname{FV}_{k+1}(r)}{r^{\frac{k+1}{k}}} \geq 2\varepsilon_0 > 0$$

for some $\varepsilon_0 > 0$. In particular, there is a sequence $T_m \in \mathbf{I}_k(X)$ with $\partial T_m = 0$ and such that $\mathbf{M}(T_m) \to \infty$ and

$$\operatorname{Fillvol}(T_m) \geq \varepsilon_0 \mathbf{M}(T_m)^{\frac{k+1}{k}} \tag{17}$$

for every $m \in \mathbb{N}$. By Theorem 1.2 of [28], X admits an isoperimetric inequality of Euclidean type for $\mathbf{I}_k(X)$ and, if $k \geq 2$ also one for $\mathbf{I}_{k-1}(X)$. Therefore we may assume by Lemma 6.2 that

$$\operatorname{diam}(\operatorname{spt} T_m) \leq B\mathbf{M}(T_m)^{\frac{1}{k}} \tag{18}$$

and

$$\|T_m\|(B(x,r)) \geq Cr^k \tag{19}$$

for all $x \in \operatorname{spt} T_m$ and all $r \in [0, 5\delta\mathbf{M}(T_m)^{1/k}]$, where B, C, δ are constants independent of m. We set $r_m := \mathbf{M}(T_m)^{\frac{1}{k}}$ and note that $r_m \to \infty$. We choose $S_m \in \mathbf{I}_{k+1}(X)$ with $\partial S_m = T_m$ and

$$\mathbf{M}(S_m) \leq D_k[\mathbf{M}(T_m)]^{\frac{k+1}{k}}$$

and with the volume growth property of Proposition 4.2. We define a sequence of metric spaces $X_m := (X, \frac{1}{r_m}d_X)$ where d_X denotes the metric on X. Setting $Z_m := \operatorname{spt} S_m \subset X_m$ it follows directly from Proposition 4.2 and (18) and (19) that the sequence $(Z_m, \frac{1}{r_m}d_X)$ is equi-compact and equi-bounded. Therefore, by Gromov's compactness theorem there exists (after passage to a subsequence) a compact metric space (Z, d_Z) and isometric embeddings $\varphi_m : (Z_m, \frac{1}{r_m}d_X) \hookrightarrow (Z, d_Z)$ and we may assume that $\varphi_m(Z_m)$ is a Cauchy sequence with respect to the Hausdorff distance. Denote by S'_m the current S_m viewed as an element of $\mathbf{I}_{k+1}(X_m)$. Since $\mathbf{M}(\varphi_{m\#}S'_m) \leq D_k$ and $\mathbf{M}(\partial(\varphi_{m\#}S'_m)) = 1$ we may assume by the compactness and closure theorems for currents that $\varphi_{m\#}S'_m$ weakly converges to some $S \in \mathbf{I}_{k+1}(Z)$. We show that $\partial S \neq 0$. For this we choose $x_m \in \operatorname{spt} S'_m$ arbitrarily and define an auxiliary metric space Y as the disjoint union $\bigsqcup_{m=1}^{\infty} X_m$ and endow it with the metric d_Y in such a way that $d_Y|_{X_m \times X_m} = \frac{1}{r_m}d_X$ as well as

$$d_Y(y, y') = \frac{1}{r_m}d_X(y, x_m) + 3 + \frac{1}{r_{m'}}d_X(y', x_{m'})$$

whenever $y \in X_m$ and $y' \in X_{m'}$ with $m' \neq m$. It is clear that Y is 2-quasiconvex and admits a local cone type inequality for $\mathbf{I}_l(Y)$, $l = 1, \dots, k$, in the sense of [29]. Denote by T'_m the current T_m viewed as an element of $\mathbf{I}_k(Y)$ and note that $\mathbf{M}(T'_m) = 1$. We show that T'_m does not weakly converge to 0. For this, assume in the contrary that T'_m weakly converges to 0. Then by Theorem 1.4 in [29] we have $\operatorname{Fillvol}(T'_m) \to 0$. In particular, there exist $\hat{S}_m \in \mathbf{I}_{k+1}(Y)$ with $\partial\hat{S}_m = T'_m$ for all $m \in \mathbb{N}$ and such that $\mathbf{M}(\hat{S}_m) \to 0$. Of course, it is not restrictive to assume that $\operatorname{spt}\hat{S}_m \subset X_m$. Denote by $\tilde{S}_m$ the current $\hat{S}_m$ viewed as a current in X. Then $\tilde{S}_m$ satisfies $\partial\tilde{S}_m = T_m$ and

$$\frac{\mathbf{M}(\tilde{S}_m)}{r_m^{k+1}} = \mathbf{M}(\hat{S}_m) \to 0,$$

which contradicts (17). Thus, T'_m does not weakly converge to 0 and therefore there exist Lipschitz maps $f, \pi_1, \dots, \pi_k \in \operatorname{Lip}(Y)$ with f bounded and $\varepsilon > 0$ such that

$$T'_m(f, \pi_1, \dots, \pi_k) \geq \varepsilon \quad \text{for all } m \in \mathbb{N}.$$

Note that $\cup Z_m \subset Y$ is bounded so that the functions π_i are bounded on $\cup Z_m$. We define Lipschitz functions f_m and π_i^m on $\varphi_m(Z_m)$ by $f_m(z) := f(\varphi_m^{-1}(z))$ and $\pi_i^m(z) :=$

$\pi_i(\varphi_m^{-1}(z))$ for $z \in \varphi_m(Z_m)$. Here, we view φ_m^{-1} as a map from $\varphi(Z_m)$ to $Y = \sqcup_{l=1}^{\infty} X_l$ with image in $X_m \subset Y$. By McShane's extension theorem there exist extensions $\hat{f}_m, \hat{\pi}_i^m : Z \to \mathbb{R}$ of f_m and π_i^m with the same Lipschitz constants as f and π_i. By Arzelà-Ascoli theorem we may assume that $\hat{f}_m$ and $\hat{\pi}_i^m$ converge uniformly to Lipschitz maps $\hat{f}$, $\hat{\pi}_i$ on Z. Finally, we abbreviate $T''_m := \varphi_{m\#} T'_m$ and use [3, Proposition 5.1] to estimate

$$\begin{aligned}
\partial S(\hat{f}, \hat{\pi}_1, \dots, \hat{\pi}_k) &= \lim_{m\to\infty} T''_m(\hat{f}, \hat{\pi}_1, \dots, \hat{\pi}_k) \\
&= \lim_{m\to\infty} \Big[T''_m(\hat{f}_m, \hat{\pi}_1^m, \dots, \hat{\pi}_k^m) + T''_m(\hat{f} - \hat{f}_m, \hat{\pi}_1, \dots, \hat{\pi}_k) \\
&\qquad\qquad + T''_m(\hat{f}_m, \hat{\pi}_1, \dots, \hat{\pi}_k) - T''_m(\hat{f}_m, \hat{\pi}_1^m, \dots, \hat{\pi}_k^m) \Big] \\
&\geq \varepsilon - \limsup_{m\to\infty} \Big[\prod_{i=1}^{k} \operatorname{Lip}(\hat{\pi}_i) \int_Z |\hat{f} - \hat{f}_m| \, d\|T''_m\| \Big] \\
&\qquad - \limsup_{m\to\infty} \Big[\operatorname{Lip}(\hat{f}_m) \sum_{i=1}^{k} \int_Z |\hat{\pi}_i - \hat{\pi}_i^m| \, d\|T''_m\| \Big] \\
&= \varepsilon.
\end{aligned}$$

This shows that indeed $\partial S \neq 0$ and hence also $S \neq 0$. Now, fix an ultrafilter ω on $\mathbb{N}$ and denote by X_ω the asymptotic cone of the sequence $(X, \frac{1}{r_m} d_X, x_0)$. We construct a map $\psi : Z' \to X_\omega$ where $Z' := \lim_H \varphi_m(Z_m) \subset Z$ is the limit with respect to the Hausdorff distance. For $z \in Z'$ there exists $z_m \in Z_m$ such that $\varphi_m(z_m) \to z$. We set $\psi(z) := (z_m)_{m\in\mathbb{N}}$. It is straight forward to check that ψ is well-defined and an isometric embedding. Since $\operatorname{spt} S \subset Z'$ we obtain that $\psi_\# S$ is a non-zero $(k+1)$-dimensional integral current in X_ω. By Theorem 4.5 in [3] there then exists a biLipschitz map $\nu : K \subset \mathbb{R}^{k+1} \to X_\omega$ where K is measurable and of strictly positive Lebesgue measure. This is in contradiction with the hypothesis that $k \geq \operatorname{asrk} X$ and hence this completes the proof. □

The arguments in the proof above can easily be used to establish the following result.

Theorem 6.3. *Let $k \in \mathbb{N}$ and let X be a complete metric space which admits isoperimetric inequalities of Euclidean type for $\mathbf{I}_{k-1}(X)$ and $\mathbf{I}_k(X)$. If $k \geq \operatorname{asrk}(X)$ then*

$$\limsup_{r\to\infty} \frac{\operatorname{FV}_{k+1}(X, L^\infty(X), r)}{r^{\frac{k+1}{k}}} = 0.$$

Note that in contrast to the main theorem we do not assume here that X admits cone type inequalities.

7. L

In this last section we will prove the following theorem.

Theorem 7.1. *Let X be a complete quasiconvex metric space and let $k \in \mathbb{N}$. Suppose X admits isoperimetric inequalities of Euclidean type for $\mathbf{I}_m(X)$ with some constants D_m, $m = 1, \dots, k-1$. If $k < \operatorname{asrk}(X)$ then*

$$\operatorname{FR}_{k+1}(r) \geq \operatorname{FR}_{k+1}(X, L^\infty(X), r) \geq \varepsilon_k r^{\frac{1}{k}}$$

for all $r > 0$ large enough and for some $\varepsilon_k > 0$ depending only on D_m, $m = 1, \dots, k-1$.

Note that Theorem 1.4 is a consequence of the above since, by Corollary 4.3, we have

$$\operatorname{Fillrad}_{L^\infty(X)}(T) \leq C[\operatorname{Fillvol}_{L^\infty(X)}(T)]^{\frac{1}{k+1}}$$

for some constant C and all $T \in \mathbf{I}_k(X)$ with $\partial T = 0$.
In the proof of the above theorem we will need the following lemma which is a direct consequence of the metric differentiability of Lipschitz maps into metric spaces proved in [18] and [22].

Lemma 7.2. *Let Z be a metric space and $\varphi : K \to Z$ Lipschitz with $K \subset \mathbb{R}^n$ Borel measurable and such that $\mathcal{H}^n(\varphi(K)) > 0$. Then there exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ with the following property: For every $\varepsilon > 0$ and for every finite set $S \subset \mathbb{R}^n$ there exist $r > 0$ and a map $\psi : S \to Z$ such that $\psi : (S, r\|\cdot\|) \to Z$ is $(1+\varepsilon)$-biLipschitz.*

Proof of Theorem 7.1. Let $X_\omega = (X, r_m, x_m)_\omega$ be an asymptotic cone of X and $\varphi : K \subset \mathbb{R}^{k+1} \to X_\omega$ a biLipschitz map with K compact and such that $\mathcal{L}^{k+1}(K) > 0$. Let $\|\cdot\|$ be a norm on $\mathbb{R}^{k+1}$ as in Lemma 7.2 and set $V := (\mathbb{R}^{k+1}, \|\cdot\|)$. Let $\{v_1, \dots, v_{k+1}\} \subset V$ and $\{v_1^*, \dots, v_{k+1}^*\} \subset V^*$ be bases satisfying

$$\|v_i\| = 1 = \|v_i^*\| \quad \text{and} \quad v_i^*(v_j) = \delta_{ij} \quad \text{for all i,j.}$$

Let Q denote the cube $Q := \left\{\sum_{i=1}^{k+1} \lambda_i v_i : 0 \leq \lambda_i \leq 1\right\}$. For $n = 1, \dots, k+1$ denote by $A(n)$ the set of increasing functions

$$\alpha : \{1, \dots, n\} \to \{1, \dots, k+1\},$$

and by L_α the subspace generated by $\{v_{\alpha(1)}, \dots, v_{\alpha(n)}\}$ whenever $\alpha \in A(n)$. It is clear that

$$1 \leq \mu_{L_\alpha}^{m*}(v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(n)}) \leq n^{\frac{n}{2}},$$

where $\mu_{L_\alpha}^{m*}$ denotes the Gromov mass*-volume on L_α and $v_{\alpha(1)} \wedge \cdots \wedge v_{\alpha(n)}$ is the parallelepiped spanned by these vectors. (Recall that for a compact set $K \subset W$ in an l-dimensional normed space the associated integer rectifiable current $[\![K]\!] \in \mathcal{I}_l(W)$ satisfies $\|[\![K]\!]\|(A) = \mu_W^{m*}(A)$, see e.g. [31, Proposition 2.7].) Fix $m \in \mathbb{N}$ large enough (as chosen below) and let Q_n denote the n-skeleton of the cubical subdivision of Q given by

$$Q_0 := Q \cap \left\{2^{-m} \sum_{i=1}^{k+1} \delta_i v_i : \delta_i \in \mathbb{Z}\right\}$$

if $n = 0$ and

$$Q_n := \{\sigma = z + 2^{-m}(L_\alpha \cap Q) : z \in Q_0, \alpha \in A(n), \sigma \subset Q\}$$

if $n \in \{1, \dots, k+1\}$. We furthermore set $\partial Q_n := \{\sigma \in Q_n : \sigma \subset \partial Q\}$. Let $s > 0$ and $\varepsilon > 0$ be arbitrary. By Lemma 7.2 there exists an $r > 0$ and a $(1+\varepsilon)$-biLipschitz map $\hat{\psi} : (\partial Q_0, r\|\cdot\|) \to X_\omega$. The definition of the asymptotic cone and the fact that ∂Q_0 is a finite set imply the existence of $s' \geq \max\{s, 2^{m+2}\}$ and a $(1+2\varepsilon)$-biLipschitz map $\psi : (\partial Q_0, s'\|\cdot\|) \to X$. We write $V' := (\mathbb{R}^{k+1}, s'\|\cdot\|)$ and note that by McShane's extension theorem, there exists a $(1+2\varepsilon)(k+1)$-Lipschitz extension $\eta : L^\infty(X) \to V'$ of the map $\psi^{-1} : \psi(\partial Q_0) \to V'$. In the following, we regard Q, Q_n, and ∂Q_n as subsets of V'. We associate with ∂Q_n the additive subgroup

$$G_n = \left\{T \in \mathbf{I}_n(V') : T = \sum c_i [\![\sigma_i]\!], c_i \in \mathbb{Z}, \sigma_i \in \partial Q_n\right\} \subset \mathbf{I}_n(V').$$

We now construct homomorphisms $\Lambda_n : G_n \to \mathbf{I}_n(X)$ and $\Gamma_n : G_n \to \mathbf{I}_{n+1}(V')$ for $n = 0, 1, \dots, k$, with the property that for all $T \in G_n$

(i) $\partial \circ \Lambda_n = \Lambda_{n-1} \circ \partial$ whenever $n \geq 1$
(ii) $\mathbf{M}(\Lambda_n(T)) \leq C'_n \mathbf{M}(T)$
(iii) $\partial \Gamma_n(T) = T - \eta_\# \Lambda_n(T) - \Gamma_{n-1}(\partial T)$ whenever $n \geq 1$
(iv) $\mathbf{M}(\Gamma_n(T)) \leq 2^{-m} D'_n s' \mathbf{M}(T)$
(v) $\operatorname{spt}(\eta_\# \Lambda_n(T)) \subset B(\operatorname{spt} T, 2^{-m} E_n s')$
(vi) $\operatorname{spt}(\Gamma_n(T)) \subset B(\operatorname{spt} T, 2^{-m} E'_n s')$.

Here, C'_n, D'_n, E_n, E'_n are constants depending only on the quasiconvexity constant D_0 and on $D_1, \dots, D_n$. For $n = 0$ we simply set $\Lambda_0([\![x]\!]) := \psi_\# [\![x]\!]$ for each $x \in \partial Q_0$ and extend Λ_0 to G_0 as a homomorphism. Set furthermore $\Gamma_0 := 0$ and note that for $n = 0$ the above properties are satisfied with $C'_0 := 1, D'_0 := 0, E_0 := 0, E'_0 := 0$. Suppose now that Λ_{n-1} and Γ_{n-1} have been defined for some $n \in \{1, \dots, k\}$ and that they have the properties listed above. In order to define Λ_n let $\sigma \in \partial Q_n$ and note that

$$(2^{-m} s')^n \leq \mathbf{M}([\![\sigma]\!]) \leq n^{\frac{n}{2}} (2^{-m} s')^n.$$

If $n = 1$ there exists a Lipschitz curve of length at most $(1+2\varepsilon) D_0 2^{-m} s'$ connecting the points $\psi(\partial\sigma)$. This gives rise to an $S \in \mathbf{I}_1(X)$ with $\partial S = \Lambda_0(\partial [\![\sigma]\!])$ which satisfies

$$\mathbf{M}(S) \leq (1+2\varepsilon) D_0 2^{-m} s' \leq (1+2\varepsilon) D_0 \mathbf{M}([\![\sigma]\!])$$

and

$$\operatorname{spt}(\eta_\# S) \subset B(\sigma, (1+2\varepsilon)^2 (k+1) D_0 2^{-m} s').$$

We define $\Lambda_1([\![\sigma]\!]) := S$. Clearly, Λ_1 satisfy properties (i) and (ii) with $C'_1 := (1+2\varepsilon) D_0$ and (v) with $E_1 := (1+2\varepsilon)^2 (k+1) D_0$. If $n \geq 2$ then

$$\partial \Lambda_{n-1}(\partial [\![\sigma]\!]) = \Lambda_{n-2}\left(\partial^2 [\![\sigma]\!]\right) = 0$$

and thus the isoperimetric inequality for $\mathbf{I}_{n-1}(X)$ and Corollary 4.3 imply the existence of an $S \in \mathbf{I}_n(X)$ such that $\partial S = \Lambda_{n-1}(\partial [\![\sigma]\!])$ and

$$\mathbf{M}(S) \leq D_{n-1} \left[\mathbf{M}(\Lambda_{n-1}(\partial [\![\sigma]\!]))\right]^{\frac{n}{n-1}} \leq C'_n (2^{-m} s')^n \leq C'_n \mathbf{M}([\![\sigma]\!]),$$

where $C'_n := D_{n-1}\left[2n(n-1)^{\frac{n-1}{2}} C'_{n-1}\right]^{\frac{n}{n-1}}$, and

$$\operatorname{spt}(\eta_\# S) \subset B\left(\sigma, \left[E_{n-1} + \mu'(1+2\varepsilon)(k+1)(n-1)^{\frac{1}{2}} (2nC'_n)^{\frac{1}{n-1}}\right] 2^{-m} s'\right).$$

We set $\Lambda_n([\![\sigma]\!]) := S$ and extend Λ_n to G_n linearly and note that properties (i), (ii) and (v) are satisfied (after the obvious choice of E_n). This completes the construction of Λ_n. In order to define Γ_n, $n \geq 1$, let again $\sigma \in \partial Q_n$ be arbitrary. Setting

$$T' := [\![\sigma]\!] - \eta_\# \Lambda_n\left([\![\sigma]\!]\right) - \Gamma_{n-1}\left(\partial[\![\sigma]\!]\right)$$

one easily checks that

$$\partial T' = \partial[\![\sigma]\!] - \eta_\#\left(\partial\Lambda_n\left([\![\sigma]\!]\right)\right) - \partial\Gamma_{n-1}\left(\partial[\![\sigma]\!]\right) = 0 \quad \text{and} \quad \mathbf{M}(T') \leq D''\mathbf{M}([\![\sigma]\!]),$$

where $D'' := \left\{1 + [(1+2\varepsilon)(k+1)]^n C'_n + 2n(n-1)^{\frac{n-1}{2}} D'_{n-1}\right\}$. Using the isoperimetric inequality for $\mathbf{I}_n(V')$ and Corollary 4.3 we find an $S \in \mathbf{I}_{n+1}(V')$ with $\partial S = T'$ and

$$\mathbf{M}(S) \leq \tilde{D}_n[\mathbf{M}(T')]^{\frac{n+1}{n}} \leq 2^{-m}\tilde{D}_n(D'')^{\frac{n+1}{n}} n^{\frac{n+1}{n}} s'\mathbf{M}([\![\sigma]\!])$$

and

$$\operatorname{spt}(\eta_\# S) \subset B\left(\sigma, \left[E_n + E'_{n-1} + \tilde{\mu}'(D'')^{\frac{1}{n}}\sqrt{n}\right] 2^{-m} s'\right),$$

where $\tilde{D}_n$ is the isoperimetric constant for $\mathbf{I}_n(V')$ and $\tilde{\mu}'$ is the constant from Corollary 4.3 for V'. We define $\Gamma_n\left([\![\sigma]\!]\right) := S$ and extend it to G_n linearly. Clearly, Γ_n satisfies the properties (iii), (iv) and (vi) with suitable choices of D'_n and E'_n. This concludes the construction of the homomorphisms Λ_n and Γ_n for $n = 0, 1, \dots, k$. Let now $[\![Q]\!] \in \mathbf{I}_{k+1}(V')$ be the integral current associated with Q endowed with the orientation $v_1 \wedge \cdots \wedge v_{k+1}$ and set $T := \Lambda_k\left(\partial[\![Q]\!]\right)$. By property (ii) we have

$$\mathbf{M}(T) \leq 2(k+1)C'_k k^{\frac{k}{2}}(s')^k.$$

Given an $S \in \mathbf{I}_{k+1}(L^\infty(X))$ with $\partial S = T$ we compute

$$\begin{aligned}
\partial\left(\eta_\# S + \Gamma_k(\partial[\![Q]\!])\right) &= \eta_\#(\partial S) + \partial(\Gamma_k(\partial[\![Q]\!])) \\
&= \eta_\#\Lambda_k(\partial[\![Q]\!]) + \partial(\Gamma_k(\partial[\![Q]\!])) \\
&= \partial[\![Q]\!],
\end{aligned}$$

where the last equality follows from property (iii). We therefore obtain

$$\mathbf{M}(\eta_\# S) \geq \mathbf{M}([\![Q]\!]) - \mathbf{M}(\Gamma_k(\partial[\![Q]\!])) \geq \left[1 - 2^{1-m}(k+1)k^{\frac{k}{2}}D'_k\right](s')^{k+1}$$

and since S was arbitrary we conclude

$$\operatorname{Fillvol}_{L^\infty(X)}(T) \geq \frac{1 - 2^{1-m}(k+1)k^{\frac{k}{2}}D'_k}{[(1+2\varepsilon)(k+1)]^{k+1}}(s')^{k+1}.$$

From this and the isoperimetric inequality for $\mathbf{I}_k(X)$ it also follows that

$$\left(\frac{1 - 2^{1-m}(k+1)k^{\frac{k}{2}}D'_k}{D_k[(1+2\varepsilon)(k+1)]^{k+1}}\right)^{\frac{k}{k+1}} \cdot (s')^k \leq \mathbf{M}(T) \leq 2(k+1)k^{\frac{k}{2}}C'_k(s')^k.$$

Furthermore, by the choice of $\{v_1, \dots, v_{k+1}\}$, we have

$$\operatorname{Fillrad}(\partial[\![Q]\!]) \geq As'$$

for some constant $A > 0$ only depending on k. We conclude

$$\mathrm{Fillrad}_{L^\infty(X)}(T) \geq \frac{A - E'_k 2^{-m}}{(1+2\varepsilon)(k+1)} s'.$$

Choose now $m \in \mathbb{N}$ sufficiently large to conclude the proof of the theorem. □

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