Mechanics I
Homework Apr. 24th; due May 8th

1. Show that the phase flow $g^t$ of a Hamiltonian system is a group.

2. Consider a pendulum with mass $m = 1$, length $l = 4$, and gravitational acceleration $g = 1$. Sketch the $(\theta, \dot{\theta})$-phase plane as well as the image of the disk $\theta^2 + (\dot{\theta} - \frac{1}{2})^2 < \frac{1}{4}$ under the phase flow $g^t$ for the time $t = \pi$. What is the relation between the areas of the disk and its image? How large is the area of the image in the $(\theta, p)$-phase plane, where $p$ is the momentum associated with $\theta$?

3. For a linear system $\dot{x} = A(t) x, x(t) \in \mathbb{R}^n, A(t) \in \mathbb{R}^{n \times n}$, let $x_1(t), \ldots, x_n(t)$ denote $n$ linearly independent solutions and introduce the Wronskian $W(t) = \det (x_1(t)|\ldots|x_n(t))$. Prove Liouville’s formula

$$W(t) = W(0) e^{ \int_0^t \text{tr} A(\tau) \, d\tau}.$$  

(Hint: The proof can proceed similarly to the proof of Liouville’s theorem.)

4. Show that in a Hamiltonian system it is impossible to have asymptotically stable equilibrium points or asymptotically stable limit cycles in the phase space (using Liouville’s theorem).

5. Consider the unit circle $S^1$ and let $g$ be rotation by an angle $\alpha$. Using Poincaré’s recurrence theorem, show that if $\frac{\alpha}{2\pi}$ is irrational, the set of points $g^k x, k \in \mathbb{N}$, is dense on $S^1$ for any $x \in S^1$.

6. Solve Burgers’ equation (which occurs e.g. in modeling gas dynamics and traffic flow),

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$

for initial data $u_0(x) = \max(0, 1 - |x|)$ at time $t = 0$, using the method of characteristics. When does the solution break down?

7. Fermat’s principle states that a light ray travelling from one point to another always takes that path which needs the least time. The speed of light in vacuum is given by $c$, the speed in a medium by $v = c/n$ for the refractive index $n$ of the medium. Of course, $n$ may depend on the spatial location $x$.

- Write down a Lagrangian $L(t, x, \dot{x})$ and the corresponding action functional for the motion of a light ray.
- Assume, the light ray is never orthogonal to the $x_3$-axis and travels in the direction of positive $x_3$-values. Do a coordinate transform in which the time $t$ is replaced by the $x_3$-coordinate (this corresponds just to reparameterizing the path) and write down the Lagrangian and action for this new time variable.
• Assume the half-space \( \{ x \in \mathbb{R}^3 : x_3 < 0 \} \) to be occupied by a material with refractive index \( n_1 \) and the complement by a material with index \( n_2 \). Consider the light ray from \((0, 0, -1)\) to \((2, 0, 1)\). Derive an equation which describes the position \((x_1, x_2, 0)\) of the light ray at the material interface.

• Derive Snell’s law of refraction: At a planar interface between two media with refractive indices \( n_1 \) and \( n_2 \), respectively, \( \frac{\sin \theta_1}{\sin \theta_2} = \frac{n_2}{n_1} \) for the angles \( \theta_1 \) and \( \theta_2 \) between the light ray and the normal to the interface in medium 1 and 2, respectively.

• Give an example where the path of a light ray (i.e. the extremal of the action) is not unique. (Hint: Think of lenses. This phenomenon is also used in astronomy: For instance, around a black hole the space-time-metric is distorted so that the black hole acts like a lens and one can thus see stars behind the black hole multiple times.)