Mechanics
- notes online: http://www.cime.nyu.edu/~wirth/mechanics.html
- weekly homework online (bring to lecture)
  - basic outline: - elasticity (~7 weeks)
    - classical mechanics (~4 weeks)
    - statistical mechanics (~3 weeks)

Elasticity
- standard references:
  - Landau, Lifshitz: Theory of Elasticity
  - Erdogan, Hughes: Mathematical Foundations of Elasticity
  - Ciarlet: Mathematical Elasticity
- brief existence theory:
  - Pedregal: Variational Methods in Nonlinear Elasticity
- many applications/examples:
  - Howells, Korszynski, Ockendon: Applied Solid Mechanics
  - Antman: Nonlinear Problems of Elasticity

Elasticity describes the deformation of solid materials and the associated forces.
Basic questions: How does solid deform under a load? What are the internal material forces?
- noninear theory (large deformations)
- linearized theory (small deformations)

Brief history (from my notes during a lecture by John Ball)
1678 Hook's law
1705 Jacques Bernoulli
1742 Daniel Bernoulli { Elastic (elastic root)
1744 Leonard Euler
1821 Navier, special case of linear elasticity via molecular model
1822 Cauchy: stress, nonlinear & linear elasticity

Key elements
- kinematics: description of (the geometry of) motion; $x(t)$, $v(t) = \text{motion}$
- kinetics/dynamics: description of forces and their relation to motion; $F(x) = \text{force}$
- statics: description of static equilibrium

Associated concepts:
- strain (local description of material deformation)
- stress (local description of material forces)
- balance laws (conservation of mass, linear & angular momentum)
- constitutive laws (relation between stress, strain, material property)

We start with kinematics, including the description of strain.

Eulerian/spatial coordinates: Fix a point $x$ in space and study how physical quantities at $x$ change over time (e.g. velocity $v(x,t)$ or density $\rho(x,t)$). Over time, $x$ is occupied by different particles. Often used in fluid dynamics, because resulting PDEs have simple form, but difficult to use with free boundaries; sometimes occupied by water, sometimes by air
Lagrangian/material coordinates: Fix a particle and track it over time. Particles are labeled by their position \( x \) in a reference configuration (often initial or unloaded configuration), but in fact need not be occupied at any time.

This is what we use in (nonlinear) elasticity, since free boundaries are handled automatically (in linearized elasticity one assumes so small deformations that a particle position stays approximate so that Lagrangian and Eulerian coords are identical).

In the reference configuration we assume the solid to occupy a sufficiently smooth, open connected region \( \Omega \subset \mathbb{R}^3 \) (a "domain").

\[ \text{e.g. particle position } y : \mathbb{R} \times [0,1] \to \mathbb{R}^3 \]
or material density \( \rho : \mathbb{R} \times [0,1] \to \mathbb{R}^3 \)

\[ \text{reason for choosing } \Omega \text{ open: contact possible} \]

\[ \text{Strain} \]

Deformation gradient: \( \mathbf{F} = \frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{pmatrix} \]

When \( \mathbf{F} : \Omega \to \mathbb{R}^{3 \times 3} \) a deformation gradient?

- \( \mathbf{F} \) is a gradient ("conservative vector field")
- \( \mathbf{F} \) has line integrals are path-independent
- \( \mathbf{F} \) has line integrals along closed curves are zero
- \( \text{curl} \mathbf{F} = (\text{curl} (\text{grad} \mathbf{y})) = 0 \)

\( \text{if } \Omega \text{ is simply connected} \)

Right Cauchy-Green deformation tensor: \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \quad \text{(after version is } \mathbf{B} = \mathbf{F}^T \mathbf{F}) \)

When \( \mathbf{C} : \Omega \to \mathbb{R}^{3 \times 3} \) a right Cauchy-Green def. tensor?

- \( \mathbf{F}^T \mathbf{F} \) is symmetric, psd, def. and thus a Riemannian metric on \( \mathbb{R}^3 \)
- \( 1dx_1^2 = \langle \mathbf{F}^T dx_1, dx_1 \rangle = |dx_1|^2 \)

This is \( 1dx^2 \) if \( \mathbf{F} \) is a def. grad., and then \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \) is the standard Euclidean metric in the deformed configuration, expressed in Lagrangian coordinates. Hence, \( \mathbf{C} \) is right Cauchy-Green def. tensor

\[ \Rightarrow \mathbf{C} \text{ is a flat metric} \]

- \( \Rightarrow \text{Riemann curvature tensor vanishes locally} \)
- \( \text{depends on } 1^{st} \text{ and } 2^{nd} \text{ derivatives of } \mathbf{C} \)

We assume \( y(\cdot, t) \) to be orientation-preserving, i.e. \( f = \det F(\cdot, t) > 0 \) for a.e. \( x \in \Omega \)
and invertible on \( \Omega \)

Inverse function theorem: \( f > 0 \Rightarrow y(\cdot, t) \) is locally invertible.

Global inverse theorem (Ball, 1982): Let both domains \( \Omega \subset \mathbb{R}^n \) have smooth boundary \( \partial \Omega \),

\[ \mathbf{y} \in C^1(\overline{\Omega} : \mathbb{R}^n) \] , \( \det D\mathbf{y} > 0 \) on \( \Omega \), \( \mathbf{y} \) is injective. Then \( \mathbf{y} \) is invertible on \( \mathbb{R}^n \).

(proof via Brouwer degree; weather version with \( \mathbf{y} \in W^{1,p}(\Omega ; \mathbb{R}^n) \), \( p > 1 \)

2nd Liptidae, \( y(x) \) satisfying a cone condition, \( \int_{\Omega} \text{det } F(\cdot, t) dx > 0 \) in Japan)
Analysis of material distortion and strain

Square root form: Let $C \in \mathbb{R}^{n \times n}$ be symm. pos. def. Then there is a unique symm. pos. def. $U \in \mathbb{R}^{n \times n}$ with $C = U^2$ ("$U = \sqrt{C}$").

Proof: Existence: Spectral decomposition yields $C = R^T \Lambda R$ for $R \in \text{SO}(n)$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with eigenvalues $\lambda_1, \ldots, \lambda_n > 0$.

$U = R^T \sqrt{\Lambda} R$ satisfies $C = U^2$ with $R^T = \text{diag}(1, \ldots, 1)$.

Uniqueness: Let $C = U^2 = V^2$ with $U, V$ symm. pos. def. and let $v$ be an eigenvector of $C$ with eigenvalue $\lambda$, then

$Cv = \lambda v \Rightarrow (U^2)v = (V^2)v \Rightarrow (Uv)^2 = (Vv)^2 \Rightarrow \frac{Uv}{Vv} = 1$

Thus $Uv = \pm Vv$. Likewise $Vv = \pm Uv$. Since the eigenvectors of $C$ span $\mathbb{R}^n$, $U$ and $V$ must coincide.

Polar decomposition form (Cauchy): If $\det F > 0$, then there exist unique matrices $R \in \text{SO}(n)$ and $U, V$ symm. pos. def. such that $F = RU = VR$.

Proof: Choose $U = \text{FF}^T, R = FV^{-1}, V = RU R^T (R \in \text{SO}(n))$ follows from $\det R = \det \text{det}(\text{det} W)^{1/2}$ and $RTR = (U^2)(U^2) = 0$.

Uniqueness follows since $F = RU = VR$ implies $U^2 = FT$ and $V^2 = FT$.

Note: $y(x + 2, t) = y(x, t) + f(x, t)2 + o(2)$ so that locally the deformation can be described by the deformation gradient $F$ (its nonlinearity).

A deformation $y(x) = Ux$ with $U$ symm. pos. def. (i.e. $U = Q^T \Lambda Q, \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ then $y = \lambda_1 x_1 + \cdots + \lambda_n x_n$)

by the factor $\lambda_1, \ldots, \lambda_n$.

Hence in a deformation with def. grad. $F$ can locally be interpreted as composition of a stretching $U$ and a rotation $R$ or of a rotation $R$ and a stretching $V$.

Using $U = Q^T \Lambda Q$ we can write $F = (Q^T \Lambda Q)^T$ which is the singular value decomposition of $F$, where $\lambda_1, \ldots, \lambda_n$ are the singular values of $F$.

Hence a deformation with def. grad. $F$ can locally be interpreted as a composition of a rotation $Q$, a stretching $\Lambda$ along the coordinate axes by $\lambda_1, \ldots, \lambda_n$, and a rotation $(Q^T \Lambda Q)$.

Since rotations do not involve material deformations (with internal forces and energy cost) the important distortion a formation is fully captured by $C$ or equivalently by the Lagrangian strain $E = \frac{1}{2}(C-I)$. In fact, one can reduce the representation even further to just three numbers which are invariant under composing $F$ with rotations from the left or right.

Strain invariants (fully describe amount of stretching/compression)

Principal stretches: singular values of $F$

Matrix invariants $I_1, I_2, I_3$: $\det (C - \lambda I) = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3$.

If $\lambda_1, \lambda_2, \lambda_3$ are principal stretches ($\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of $C$), then

$\det (C - \lambda I) = \det (Q^T (\lambda_1 x_1^2 - \lambda_2 x_2^2 - \lambda_3 x_3^2) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$.

$\Rightarrow I_1 = \lambda_1 + \lambda_2 + \lambda_3 = \text{tr} C$,

$\Rightarrow I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = \frac{1}{2} \left[ (\text{tr} C)^2 - \text{tr} C^2 \right]$,

$\Rightarrow I_3 = \lambda_1 \lambda_2 \lambda_3 = \text{det} C$.

Matrix invariants $\Gamma_1^2, \Gamma_2^2, \Gamma_3^2$: $\det (B - \lambda I) = -\lambda^3 + \Gamma_1 \lambda^2 - \Gamma_2 \lambda + \Gamma_3$.

$I_B = I_C$ since $+C = +B$; likewise $\Gamma_2^2 = \Gamma_1^2, \Gamma_3^2 = \Gamma_1^2$.