Poroplasticity (linearized theory)

Poroplastic theory describes the behavior of a porous elastic structure (the matrix material) immersed in fluid. It was developed to describe consolidation of soil, but can also be applied to biological tissues or rocks with porosity. First works by Karl von Terzaghi (~1925) and Maurice Biot (~1941, Columbia University).

Kinematic variables: displacement \( u \), volume fraction of fluid \( \Delta \), (relative to reference state, i.e., \( \Delta \mathbf{e} = \mathbf{e} - \mathbf{e}_0 \))

Dynamic variables: stress of total stress \( T \), fluid pressure \( p \), body force \( f \), and surface load \( f \) acting on elastic matrix

We directly look at linearized theory, i.e., assume small displacement \( u \) and \( \Delta \) as well as small changes in \( \Delta \); also we assume isotropic materials and an incompressible fluid. Finally, we only stress (hence, all this can be generalized)

Linearized theory refers to a relaxed porous matrix

Assume there exists an energy density of the poroplastic material \( W(u, \Delta) \)

\[
W(u, \Delta) = \frac{1}{2} \mathbf{D}_{122} \mathbf{e} : \mathbf{D}_{122} \mathbf{e} + \frac{1}{2} \mathbf{D}_{22} \mathbf{e} : \mathbf{D}_{22} \mathbf{e} - W_0 \mathbf{e}_0
\]

in small \( \Delta \), relaxed reference state

Balance laws: mass conservation of solid \( \nabla \cdot \mathbf{T} = 0 \) (trivial, since linearized equations considered)

Angular momentum: \( \mathbf{T} = T \mathbf{e} \)

Mass conservation of fluid: \( \frac{\partial \rho}{\partial t} \mathbf{e} + \nabla \cdot \mathbf{f} = 0 \) in \( \Omega \)

\[
\text{for fluid volume flux per normal cross-sectional area } f
\]

\[
\text{in eqn: } 0 = \frac{\partial}{\partial t} \mathbf{e} + \nabla f \mathbf{e} + \int \mathbf{f} \mathbf{e} \, dx
\]

\[
\text{boundary conditions: } \mathbf{T} = \mathbf{T}_0 \text{ on } \partial \Omega
\]

\[
\mathbf{u} = \mathbf{u}_0 \text{ on } \partial \Omega
\]

\[
\mathbf{p} = \mathbf{p}_0 \text{ on } \partial \Omega
\]

\[
\mathbf{f} = \mathbf{f}_0 \text{ on } \partial \Omega
\]

\[
\text{in eqn, } 0 = \frac{\partial}{\partial t} \mathbf{r}_w + \nabla \cdot \mathbf{f}
\]

\[
\mathbf{r}_w = \mathbf{r}_w \mathbf{e} + \mathbf{r}_w \mathbf{e}_0
\]

\[
\text{in addition: Darcy's law for porous flow: } f = -\mathbf{K} \mathbf{p} \text{ for permeability } \mathbf{K}
\]

\[
\text{Isotropy implies: } \mathbf{K} \text{ shear to the left and to the right produce same pressure change}
\]

\[
\Rightarrow H_2 = -H_2 \quad \Rightarrow H_3 = 0
\]

\[
\text{analogously: } H_1 = 0 \text{ for } \mathbf{e}^T
\]

\[
\text{also due to isotropy, } H_2 = H_{22} = H_{11} = 0 \quad \text{for orthotropic case}
\]

Typically, constitutive laws are written in the form

\[
\mathbf{T} = C \mathbf{e}(u) - \mathbf{p}
\]

\[
\Delta = \Delta(u) - \Delta_0
\]
we identify the coefficients $\lambda, \mu \in \mathbb{R}^{+\infty}, \alpha \in \mathbb{R}$:

\[
(\alpha =) \quad \Delta \mathbf{e} = \frac{1}{\rho} \alpha \mathbf{e}(\mathbf{u}) \quad \text{this is } \alpha \Rightarrow T = \frac{3}{2} E \mathbf{e}(\mathbf{u}) + \nu^2 T(\mathbf{e}(\mathbf{u})) I + \epsilon^2 I
\]

stress due to deformation pressure, contribution of solid matrix.

The last equation, the decomposition of the stress in a part due to the solid matrix and a pressure component, is called "Terzaghi's main principle." The typically used coefficients are

\[
T = 2\mu \epsilon(\mathbf{u}) + \lambda \epsilon(\mathbf{u}) I + \alpha \epsilon I
\]

\[
\Delta = \lambda \epsilon(\mathbf{u}) + (\lambda + 2\mu) \epsilon(\mathbf{u}) + \epsilon^2 I
\]

for $Biot - Willis$ parameter $\alpha$.\[Knapton's\ coefficient\ $\lambda$, \[modulus\ \kappa\ and\ Lamé\ constant\ \mu$ of the matrix structure (not of the composite material)!)

Rx: Solving the equations (A), (B), (C), (D), (E), one can see various interesting phenomena, e.g.:

that a porous solid only deforms slowly when a load is applied, since all the fluid has to be squeezed out.

A list of further interesting topics in elasticity (still active research):

- Dimension reduction (describe a 3D-solid of thickness $h$ and find out what equations govern the deformation as $h \to 0$) and resulting plate theory (or shell) (e.g. Fried, Schanz, Müller: Comm. Pure Appl. Math. 2002)

- Shape optimization (find the optimal geometry of an elastically loaded device) (e.g. Benner, Bumpy, Wirth: ESAIM COCV 2013)

- Micro-pattern formation in elastic objects (e.g. Bella, Kohn: Comm. Pure Appl. Math. 2012)

- Phase transformations and martensite (Bella, Kohn: "Microstructure of Martensites" 2003)