Classical Newtonian mechanics treat the motion of finitely many rigid objects, in the simplest case mass points (i.e., points with zero spatial extension but non-zero mass m). These objects can interact with each other (e.g., they can attract or repel each other) or the environment. “Classical” means that quantum effects do not occur. Classical mechanics can for instance be used to describe planetary motion or systems of interacting particles (e.g., to some extent molecules is a gas).

A classic: Landau/Lifshitz: Mechanics

An introductory text: Goldstein: (A Brief Introduction to Classical, Statistical and Quantum Mechanics)

More advanced: Arnold: Mathematical Methods of Classical Mechanics

Classical mechanics are based on Newton’s law of motion, an axiom from physics, which holds for any particle and in words can be expressed as “mass times acceleration equals force”, or, using the momentum $p = m v$, of a particle with mass $m > 0$ and velocity $v \in \mathbb{R}^n$.

\[
\frac{dp}{dt} = F
\]

where $t$ is time, $F \in \mathbb{R}^n$ is the force acting on the particle, which may depend on particle position, velocity, or other particles, etc.

For a set of particles with positions $q_i \in \mathbb{R}^n$, Newton’s law produces a system of $n$ first order differential equations for the $q_i$, that can be solved given initial values, such as an initial position $q_i(0)$ and velocity $v_i(0) \in \mathbb{R}^n$.

There are three different perspectives on this problem, which all shed light on different aspects:

1. ODE system as introduced above (Newton 1687: Principia Mathematica)
2. Lagrangian viewpoint: action principle: The motion is described as a high-dimensional optimization problem (Lagrange 1788: reformulation of Newton’s law in “generalized coordinates”;
   Lagrangian multiform constrained motion; action principle has longer history)
3. Hamiltonian viewpoint: volume-preserving flow in phase space (formulation by Hamilton 1833)

**Guiding examples**

PM) point mass in potential field (e.g. gravitational field)

The force is given as gradient of a fixed external potential $V$, $F = - \nabla V$, i.e.

\[
\frac{dp}{dt} = m \frac{d(\dot{q})}{dt} = m \ddot{q} = - \nabla V(q)
\]

a) gravitational field on earth: $V(q) = m g q_3 \Rightarrow \ddot{q} = -g \epsilon(0) \Rightarrow q = q_0 + \dot{q}(0) t - \frac{1}{2} \frac{\epsilon(0)}{g} t^2$

b) height $q_3$ of a ball thrown in the air:

c) earth (mass $m$) in sun’s gravitational field (assuming sun’s mass M to be stationary):

\[
V = - m \frac{M}{r^2} \Rightarrow \ddot{r} = - r \frac{M}{r^4} \quad \text{(acceleration towards center)}
\]

in polar coord. (restricting to 2D): $\dot{\theta}^2 \left( r^2 \dot{r} + 2 r \dot{r} \right) - r \ddot{r} \left( r \dot{\theta}^2 - \dot{r} \right) = \frac{M}{r^2}$

Example solution:

\[
(\theta(0) = \theta_0, \dot{\theta}(0) = 0) \Rightarrow (\theta(t)) = \frac{\theta_0}{\sqrt{1 + \frac{\theta_0^2}{g^2}}} \left( \sin(t) \right)
\]

P) pendulum:

\[
\begin{align*}
\ddot{\theta} + \frac{g}{l} \sin \theta &= 0
\end{align*}
\]

for small $\theta$, $\sin \theta \approx \theta$, thus $\theta(t) \approx \theta_0 t + \frac{1}{2} \omega_0 (t_0 - t)$

MR) multi-body problem: attracting particles (e.g. planets)

\[
V(q_i) = - y m; \sum_{j \neq i} \frac{m_i}{|q_i - q_j|} \Rightarrow \ddot{q}_i = \nabla \sum_{j \neq i} \frac{m_i}{|q_i - q_j|}
\]

no closed form solution for $\geq 3$ particles (sun, earth, moon)
Genetic & potential energy. Lagrangian, Hamiltonian.

- Often we consider the case that the force only depends on position \( q \) and is the gradient of a potential energy \( V \), \( F = -\frac{\partial}{\partial q} V(q) \). Also, the momentum \( p_i \) of each particle is typically given by \( m_i \frac{dq_i}{dt} = \frac{\partial}{\partial q_i} F(q) \) for the so-called kinetic energy \( \frac{1}{2} \sum \frac{p_i^2}{m_i} \). The total physical energy then is \( H = \frac{1}{2} \sum \frac{p_i^2}{m_i} + V(q) \). Note that the potential energy is always relative to a reference state, i.e., \( V + \text{const.} \) is also a valid potential energy.

- We then introduce the Hamiltonian \( H = T + V \) and the Lagrangian \( L = T - V \). Obviously, \( p_i = \frac{\partial L}{\partial \dot{q}_i} \) and \( F_i = \frac{\partial H}{\partial q_i} \).

In a more abstract setting we assume that at any time \( t \) our system is specified by a position vector or generalized coordinate \( \mathbf{q}(t) \in \mathbb{R}^n \) which can, e.g., be composed of the particle positions \( \mathbf{q} = (q_1, q_2, \ldots, q_n) \) and the velocity vector \( \dot{\mathbf{q}} \). Furthermore, we assume that \( \mathbf{q} \) is a Lagrangian for \( L(t, \mathbf{q}, \dot{\mathbf{q}}) \); i.e., the momentum vector of the system is given by \( \mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} \) and the force vector by \( \mathbf{F} = \frac{\partial H}{\partial \mathbf{q}} \). (X) then reads

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial q_i} \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial \dot{q}_i}
\]

The Hamiltonian is then defined as the Legendre-Fenchel dual of \( L \) in the \( (q, \dot{q}) \)-argument,

\[
H(q, \dot{q}) = \sup_{\dot{q}} \dot{q} \cdot p - L(q, \dot{q})
\]

and it is interpreted as the total energy of the system.

**Example:**

- \( P(\dot{q}) \)
  
  \[
  T = \frac{1}{2} \sum m_i \dot{q}_i^2 \quad V = \sum m_i g \dot{q}_i \quad L(\dot{q}, q) = \frac{1}{2} \sum m_i \dot{q}_i^2 - m_i g q_i \quad H(\dot{q}, \dot{q}, q) = \frac{1}{2} \sum m_i \dot{q}_i^2 + m_i g q_i
  \]

- \( P(\theta) \)
  
  \[
  T = \frac{1}{2} \sum \frac{1}{2} \beta \dot{\theta}_i^2 \quad V = -mg \theta \quad L(\dot{\theta}, \theta) = \frac{1}{2} \sum \frac{1}{2} \beta \dot{\theta}_i^2 + mg \theta \quad H(\dot{\theta}, \theta) = \frac{T}{2} \sum \frac{1}{2} \beta \dot{\theta}_i^2 + \frac{1}{2} \beta \theta_i^2
  \]

**Note:** Had we chosen other generalized coordinates, then \( q = \theta \), e.g., \( q = \theta \); then it would have got a different Lagrangian, momentum, and Hamiltonian. However, we will see that (XX) holds for any choice of coordinates.

**Example:**

- \( M(\dot{q}, \mathbf{q}) \)
  
  \[
  T = \frac{1}{2} \sum \frac{1}{2} \beta \dot{q}_i^2 \quad V = -\sum \frac{1}{2} \beta \dot{q}_i^2 \quad L(\dot{q}, \mathbf{q}) = \frac{1}{2} \sum \frac{1}{2} \beta \dot{q}_i^2 - \sum \frac{1}{2} \beta \dot{q}_i^2 \quad H(\dot{q}, \mathbf{q}) = \frac{1}{2} \sum \dot{q}_i^2 - \frac{1}{2} \beta \mathbf{q}^2
  \]

We will hence always assume \( L \) to be a convex in \( (q, \dot{q}) \)-argument

- \( \text{differentiable} \)
- \( \text{suplinear growth in } q \)

\[ \Rightarrow H(q, \dot{q}, p) = \max q \cdot p - L(q, \dot{q}) \text{ is differentiable} \]

**Lemma:** Let \( f(x, y) \) and \( f^*(p, y) = \sup_x x \cdot p - f(x, y) \) be convex in \( x, p \), resp. and differentiable, then

\[
\frac{\partial f}{\partial x} = y \quad \text{and} \quad \frac{\partial f^*}{\partial p} = -\left( \frac{\partial f}{\partial y} \right)_x \],

where the relation between the coordinates is determined by \( p = \frac{\partial}{\partial y} (x, y) \).

**Proof:**

\[
f^*(p, y) = p \cdot x - f(x, y) \text{ where } x = x(p, y) \text{ satisfies } \frac{\partial f}{\partial y}(x(p, y), y) = \frac{p}{\partial y} \text{ (uniquely solvable due to convexity).}

Now \( \frac{\partial f}{\partial p} = \frac{\partial}{\partial y} (p \cdot x - f(x, y)) = x, \) and \( \frac{\partial f}{\partial y} = -\frac{\partial}{\partial y} (p \cdot x + \frac{p}{\partial y} (p \cdot x - f(x, y))) = -\frac{p}{\partial y} \).

**Lemma:**

\[
L(t, \dot{q}, \dot{q}) = \sup_{\dot{q}} \dot{q} \cdot p - H(t, \dot{q}, p)
\]

**Proof:** (a) by properties of Legendre transform.

(b) \( H = p \cdot \dot{q} - L(t, q, \dot{q}) \) with \( p = \frac{\partial}{\partial \dot{q}} (L(t, \dot{q}, \dot{q})) \)

\[
- \dot{p} = \frac{\partial}{\partial q} (\frac{\partial L}{\partial \dot{q}}) = \frac{\partial}{\partial q} (\frac{p}{\partial \dot{q}}) = \frac{p}{\partial q} = \frac{2}{\partial q} \quad \text{by lemma}
\]

\[
\frac{\partial J}{\partial p} = \frac{\partial}{\partial \dot{q}} \frac{\partial L}{\partial q} \quad \text{by Lemma}
\]