Gaussian fluctuations for random walks in random mixing environments

Francis Comets* Ofer Zeitouni†

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Abstract

We consider a class of ballistic, multidimensional random walks in random environments where the environment satisfies appropriate mixing conditions. Continuing our previous work [2] for the law of large numbers, we prove here that the fluctuations are gaussian when the environment is Gibbsian satisfying the "strong mixing condition" of Dobrushin and Shlosman and the mixing rate is large enough to balance moments of some random times depending on the path. Under appropriate assumptions the CLT applies in both non-nestling and nestling cases, and trivialy in the case of finite-dependent environments with "strong enough bias". Our proof makes use of the asymptotic regeneration scheme introduced in [2]. When the environment is only weakly mixing, we can only prove that if the fluctuations are diffusive then they are necessarily Gaussian.

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1 Introduction and Main Statements

1.1 Introduction

Fix an integer $d > 1$, let S denote the set of 2d-dimensional probability vectors, and set $\Omega = S^{\mathbb{Z}^d}$. We consider all $\omega \in \Omega$ as an "environment" for the random walk that we define below in (1.1), and

*Université Paris 7 — Denis Diderot, Mathématiques, case 7012, 2 place Jussieu, 752151 Paris Cedex 05, France. Partially supported by CNRS, UMR 7599 "Probabilités et Modèles aléatoires".

†Department of Mathematics, University of Minnesota, Minneapolis 55455, USA, on leave from the Departments of Electrical Engineering and of Mathematics, Technion, Haifa 32000, Israel. Partially supported by NSF grant number DMS-0302230.

we denote by $\omega(z,\cdot) = \{\omega(z,z+e)\}_{e\in\mathbb{Z}^d,|e|=1}$ the coordinate of $\omega \in \Omega$ corresponding to $z \in \mathbb{Z}^d$. The random walk in the environment ω started at $z \in \mathbb{Z}^d$ is the Markov Chain $\{X_n\} = \{X_n; n \geq 0\}$ with state space $\mathbb{Z}^d$ such that $X_0 \equiv z$ and

$$P_\omega^z(X_{n+1} = x+e | X_n = x) = \omega(x, x+e)\,, \quad e \in \mathbb{Z}^d, |e| = 1\,. \tag{1.1}$$

Let P be a probability measure on Ω, stationary and ergodic with respect to the shifts in $\mathbb{Z}^d$. We denote by $\mathbb{P}^z = P \otimes P_\omega^z$ the joint law on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ of $\{X_n\}_n$ and ω. The process $\{X_n\}$ under $\mathbb{P}^z$, is called the random walk in random environment (RWRE). We will denote by $\mathbb{E}^z = E_{\mathbb{P}^z}, E_\omega^z = E_{P_\omega^z}$ the expectations corresponding to $\mathbb{P}^z, P_\omega^z$, respectively.

Much is known about the RWRE when $d = 1$, see [16] for a recent review, including a discussion of laws of large numbers and central limit theorems for product and non-product measures P. See also [11] for recent stable limit results with $d = 1$ and non-product environments. In dimension $d > 1$, when P is a product measure and in the ballistic regime, i.e. when there exists a deterministic direction $\ell \in S^{d-1}$ such that $\limsup X_n \cdot \ell / n = v_\ell > 0$, the law of large numbers was first derived in the seminal paper [14] using a regenerative scheme. In the same context of P being a product measure, the central limit theorem for $\{X_n\}$ was obtained in [13], assuming uniform ellipticity and Kalikow's condition, using this regenerative scheme. Further development (in the ballistic case with P a product measure) can be found in [15].

In the case of dependent environment, laws of large numbers have been obtained in [7], [8], [12], with a rather mild dependence structure. More realistic dependence structures – including Gibbs measures in the mixing regime – were considered in [10] and [2], where the law of large numbers is proved. In [10], the author uses the approach of environment viewed from the point of view of the particle, while in our previous [2] we introduce a coupling method to find an asymptotic regenerative scheme.

Our goal in this paper is to adapt this latter technique to derive central limit theorems for the RWRE when P is not a product measure. This provides then, to our knowledge, the first example of RWRE's in a dependent environment which do not exhibit finite range dependence, for which CLT type statements hold.

1.2 Some Assumptions: Mixing, Ellipticity, Drift

In the sequel, we fix an $\ell \in \mathbb{R}^d \setminus \{0\}$ such that ℓ has *integer coordinates.* With $\text{sign}(0) = 0$ and $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{Z}^d$, let

$$\mathcal{E}_{\bar{\varepsilon}} = \{\text{sign}(\ell_i) e_i\}_{i=1}^d \setminus \{0\}\,. \tag{1.2}$$

Define the cone of vertex $x \in \mathbb{R}^d$, direction ℓ and angle $\cos^{-1}(\zeta), \zeta \in (0,1)$, by

$$C(x,\ell,\zeta) = \{y \in \mathbb{R}^d\ ;\ (y-x)\cdot\ell \geq\ \zeta|y-x||\ell|\}\,. \tag{1.3}$$

We also need in the sequel the *truncated* ℓ cones defined as

$$C(x,\ell,\zeta,M) = \{y \in \mathbb{R}^d \; ; \; y \in C(x,\ell,\zeta), (y-x)\cdot \ell \le M\};. \tag{1.4}$$

In [2], we made the following two assumptions on the environment:

Assumption 1.5

($\mathcal{A}1$) *P is stationary and ergodic, and satisfies the following mixing condition on ℓ-cones: for all positive ζ small enough there exists a function $\phi(r) \underset{r\to\infty}{\to} 0$ such that any two events A, B with $P(A) > 0$, $A \in \sigma\{\omega_z; z\cdot\ell \le 0\}$ and $B \in \sigma\{\omega_z; z \in C(r\ell,\ell,\zeta)\}$ it holds that*

$$\left| \frac{P(A\cap B)}{P(A)} - P(B) \right| \le \phi(r|\ell|)\,.$$

($\mathcal{A}2$) *P is elliptic and uniformly elliptic with respect to ℓ: $P(\omega(0,e) > 0; |e| = 1) = 1$, and there exists a $\kappa > 0$ such that*

$$P(\min_{e\in\mathcal{E}_{\bar{\varepsilon}}} \omega(0,e) \ge 2\kappa) = 1\,.$$

As described in [2], condition $\mathcal{A}1$ is satisfied for a class of Gibbs random field satisfying the so called *weak mixing* condition of Dobrushin and Shlosman. For the strong CLT results, we will need a stronger notion of mixing, based on Dobrushin-Shlosman's *strong mixing* condition. We introduce next this notion, starting with the

Definition 1.6 *Let $k \ge 1$, and let $\partial\Lambda^k = \{z \in \Lambda^c; \mathrm{dist}\;(z,\Lambda) \le k\}$ be the k-boundary of $\Lambda \subset \mathbb{Z}^d$. (*dist *and $|.|$ both denote the Euclidean distance). A random field P is* k-**Markov** *if there exists a family π of transition kernels — called specification — $\pi_\Lambda = \pi_\Lambda(\prod_{y\in\Lambda} d\omega_y | \,\mathcal{F}_{\partial\Lambda})$ for finite $\Lambda \subset \mathbb{Z}^d$ such that*

$$P\big((\omega_x)_{x\in\Lambda} \in \cdot \mid \mathcal{F}_{\Lambda^c}\big) = \pi_\Lambda\big(\;\cdot\; \mid \mathcal{F}_{\partial\Lambda}\big), \quad P-a.s. \tag{1.7}$$

In addition, a k-Markov field P is called **strong-mixing** *if there exist constants $\gamma > 0$, $C < \infty$ such that for all finite subsets $V \subset \Lambda \subset \mathbb{Z}^d$ and all $y \in \Lambda^c$,*

$$\sup\big\{\|\pi_\Lambda(\cdot\mid\omega) - \pi_\Lambda(\cdot\mid\omega')\|_V;\; \omega,\omega' \in S^{\Lambda^c}, \omega_x = \omega'_x\, \forall x \ne y\}\big\} \le C \sum_{z\in\partial V^k} \exp(-\gamma|z-y|)\,, \tag{1.8}$$

with $\|.\|_V = \|.\|_{\mathrm{Var},V}$ the variational norm on V, $\|\mu-\nu\|_V = \sup\{\mu(A) - \nu(A); A \subset \sigma((\omega_x)_{x\in V})\}$.

The strong-mixing property holds for environments produced by a Gibbsian particle system at equilibrium in the uniqueness regime at high enough temperatures, see [3, 9]. Strong-mixing environments are weak-mixing, and by [2, Proposition 4.4], they are mixing on cones in the sense of Assumption $\mathcal{A}1$. Sumarizing this, we have $(\mathcal{A}1') \Rightarrow (\mathcal{A}1)$, where we set

Assumption 1.9

($\mathcal{A}1'$) *P is a Gibbs, strong-mixing, Markov field.*

We will also need some conditions on the environment ensuring the ballistic nature of the walk. Let U be a finite, connected subset of $\mathbb{Z}^d$, with $0 \in U$, let $\mathcal{F}_{U^c} = \sigma\{\omega_z : z \notin U\}$, and define on $U \cup \partial U$ an auxiliary Markov chain with transition probabilities

$$\hat{P}_U(x, x+e) = \begin{cases} \dfrac{\mathbb{E}^o\left[\sum\limits_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n=x\}}\omega(x,x+e)|\mathcal{F}_{U^c}\right]}{\mathbb{E}^o\left[\sum\limits_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n=x\}}|\mathcal{F}_{U^c}\right]}, & x \in U, |e| = 1 \\ 1 & x \in \partial U, e = 0 \end{cases} \tag{1.10}$$

where $T_{U^c} = \min\{n \geq 0 : X_n \in \partial U\}$. This chain is known as Kalikow's Markov chain [6]. We will denote by $\hat{d}_U(x) = \sum_{|e|=1} e\hat{P}_U(x, x+e)$ the Kalikow drift, and $d(x,\omega) = \sum_{|e|=1} e\omega(x, x+e)$ the RWRE's drift at x.

In addition to $\mathcal{A}1$ and $\mathcal{A}2$, we will assume one of the two following drift conditions, which ensure a ballistic behavior for the walk:

Assumption 1.11

($\mathcal{A}3$) **Kalikow's condition**: *There exists a $\delta(\ell) > 0$ deterministic such that*

$$\inf_{U, x\in U} \hat{d}_U(x) \cdot \ell \geq \delta(\ell),\ P-a.s..$$

(The infimum is taken over all connected finite subsets of $\mathbb{Z}^d$ containing 0.)

($\mathcal{A}4$) **Non-nestling condition**: *There exists a $\delta(\ell) > 0$ such that*

$$d(x,\omega) \cdot \ell \geq \delta(\ell)\,, P-a.s.$$

Clearly, ($\mathcal{A}4$) is stronger than ($\mathcal{A}3$). Both conditions imply that $\lim_{n\to\infty} X_n \cdot \ell = \infty$ $\mathbb{P}^o$-a.s.

1.3 Asymptotic Regenerative Scheme

In this section, we recall some constructions and results from [2].

First, we define the RWRE on an enlarged space, depending on the vector ℓ with integer coordinates: instead of considering the law $\mathbb{P}^o = P \otimes P^o_\omega$ on the canonical space $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$, we consider the following probability measure

$$\overline{\mathbb{P}}^o = P \otimes Q \otimes \overline{P}^o_{\omega,\varepsilon} \quad \text{on } \Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}} ,$$

with $W = \{0\} \cup \mathcal{E}_{\bar{\varepsilon}}$ and $\mathcal{E}_{\bar{\varepsilon}}$ from (1.2)): Q is a product measure, such that with $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ denoting an element of $W^{\mathbb{N}}$, $Q(\varepsilon_1 = e) = \kappa$, for $e \in \mathcal{E}_{\bar{\varepsilon}}$, while $Q(\varepsilon_1 = 0) = 1 - \kappa|\mathcal{E}_{\bar{\varepsilon}}|$. For each fixed ω, ε, $\overline{P}^o_{\omega,\varepsilon}$ is the law of the Markov chain $\{X_n\}$ with state space $\mathbb{Z}^d$, such that $X_0 = 0$ and, for every $z, e \in \mathbb{Z}^d$, $|e| = 1$,

$$\overline{P}^o_{\omega,\varepsilon}(X_{n+1} = z+e \mid X_n = z) = \mathbf{1}_{\{\varepsilon_{n+1}=e\}} + \frac{\mathbf{1}_{\{\varepsilon_{n+1}=0\}}}{1-\kappa|\mathcal{E}_{\bar{\varepsilon}}|} \left[\omega(z, z+e) - \kappa \mathbf{1}_{\{e \in \mathcal{E}_{\bar{\varepsilon}}\}}\right] . \tag{1.12}$$

The point is that, the law of $\{X_n\}$ under $Q \otimes \overline{P}^o_{\omega,\varepsilon}$ coincides with its law under P^o_ω, while its law under $\overline{\mathbb{P}}^o$ coincides with its law under $\mathbb{P}^o$.

We fix now a particular sequence of ε in $\mathcal{E}_{\bar{\varepsilon}}$ of length $|\ell|_1$ with sum equal to ℓ: for definiteness, we take $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_{|\ell|_1})$ with $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \dots \bar{\varepsilon}_{|\ell_1|} = \mathrm{sign}(\ell_1) e_1$, $\bar{\varepsilon}_{\ell_1+1} = \bar{\varepsilon}_{\ell_1+2} = \dots \bar{\varepsilon}_{|\ell_1|+|\ell_2|} = \mathrm{sign}(\ell_2) e_2$, ... $\bar{\varepsilon}_{|\ell|_1 - |\ell_d|+1} = \dots \bar{\varepsilon}_{|\ell|_1} = \mathrm{sign}(\ell_d) e_d$ .. We assume, through the whole paper, $\zeta > 0$ small enough such that

$$\bar{\varepsilon}_1, \bar{\varepsilon}_1 + \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_1 + \dots \bar{\varepsilon}_{|\ell|_1} = \ell \in C(0, \ell, \zeta) \quad , \tag{1.13}$$

and such that $(\mathcal{A}1)$ above is satisfied.

For $L \in |\ell|_1 \mathbb{N}^*$ we will denote by $\bar{\varepsilon}^{(L)}$ the vector

$$\bar{\varepsilon}^{(L)} = (\bar{\varepsilon}, \bar{\varepsilon}, \dots, \bar{\varepsilon})$$

of dimension L. Define

$$D' = \inf\{n \geq 0 : \ X_n \notin C(X_0, \ell, \zeta)\} \quad . \tag{1.14}$$

Assumption $(\mathcal{A}3)$ implies $\mathbb{P}^o(D' = \infty | \omega_x, x \cdot \ell \leq 0)$ is bounded away from 0. For all $L \in |\ell|_1 \mathbb{N}$, set $\overline{S}_0 = 0$ and, using θ_n to denote time shift, set

$$\begin{aligned} \overline{S}_1 &= \inf\Big\{n \geq L : X_{n-L} \cdot \ell > \max\{X_m \cdot \ell : \ m < n-L\}, \, (\varepsilon_{n-L}, \dots, \varepsilon_{n-1}) = \bar{\varepsilon}^{(L)}\Big\} \leq \infty \ , \\ \overline{R}_1 &= D' \circ \theta_{\bar{S}_1} + \overline{S}_1 \leq \infty. \end{aligned} \tag{1.15}$$

Note that the random times $\overline{S}_1, \overline{R}_1$, depend on both $\{X_n\}_n$ and $\{\varepsilon_n\}_n$. Define further, by induction for $k \geq 1$,

$$\begin{aligned} \overline{S}_{k+1} &= \inf\Big\{n \geq \overline{R}_k : X_{n-L} \cdot \ell > \max\{X_m \cdot \ell : \ m < n-L\}, \, (\varepsilon_{n-L}, \dots, \varepsilon_{n-1}) = \bar{\varepsilon}^{(L)}\Big\} \leq \infty \ , \\ \overline{R}_{k+1} &= D' \circ \theta_{\bar{S}_{k+1}} + \overline{S}_{k+1} \leq \infty \ , \end{aligned}$$

These variables are stopping times for the pair $\{X_n, \varepsilon_n\}_n$ (depending on L), with

$$0 = \overline{S}_0 \leq \overline{S}_1 \leq \overline{R}_1 \leq \overline{S}_2 \leq \cdots \leq \infty$$

and the inequalities are strict if the left member is finite. Also, since $X_n \cdot \ell \to_{n\to\infty} \infty$, $\overline{S}_{k+1}$ is $\overline{\mathbb{P}}^o$-a.s. finite on the set $\{\overline{R}_k < \infty\}$. Define:

$$\tau_1^{(L)} = \overline{S}_K \leq \infty\,, \quad \text{with } K = \inf\{k \geq 1 : \overline{S}_k < \infty, \overline{R}_k = \infty\} \leq \infty\,.$$

This random time $\tau_1^{(L)}$ is the first time n when the walk performs as follow: at time $n - L$ it has reached a record value in the direction $+\ell$, then it travels using the ε-sequence only up to time n, and from time n on, it does not exit the positive cone $C(X_n, \ell, \zeta)$ with vertex X_n. The advantage in considering $\tau_i^{(L)}$, is that at these times, the RWRE travels $|L|_1$ time units in the direction ℓ, **without learning any information about the environment**, allowing for decorrelation.

Under $(\mathcal{A}1, 2, 3)$, and if $\zeta \leq \delta(\ell)/(3|\ell|)$, then $\tau_1^{(L)}, \tau_2^{(L)}, \ldots$ are finite $\overline{\mathbb{P}}^o$-a.s. for large L. For $L \in |\ell|_1\mathbb{N}^*$ we define $\tau_0^{(L)} = 0$, and for $k \geq 1$,

$$\overline{\tau}_k^{(L)} = \kappa^L \left(\tau_k^{(L)} - \tau_{k-1}^{(L)}\right)\,, \qquad \overline{X}_k^{(L)} = \kappa^L \left(X_{\tau_k^{(L)}} - X_{\tau_{k-1}^{(L)}}\right)\,. \tag{1.16}$$

(A rescaling by the factor κ^L is needed in order to keep the variables $\overline{\tau}_k^{(L)}, \overline{X}_k^{(L)}$ of order 1 as $L \to \infty$.) The above random times yield an asymptotic (in the limit $L \to \infty$) regenerative structure, which can be expressed in term of the following coupling, see Section 3 in [2]:

Coupling: We can enlarge once again our probability space [and we will continue to denote by $\overline{\mathbb{P}}^o$ annealed probabilities in this larger space], where is defined the sequence $\{(\overline{\tau}_i^{(L)}, \overline{X}_i^{(L)})\}_{i\geq 1}$, in order to support also:

- a sequence $\{(\tilde{\tau}_i^{(L)}, \tilde{X}_i^{(L)}, \Delta_i^{(L)})\}_{i\geq 1}$ of i.i.d. random vectors (with values in $\kappa^L\mathbb{N}^* \times \kappa^L\mathbb{Z}^d \times \{0,1\}$) where $\Delta_i^{(L)} \in \{0,1\}$ is such that

$$\overline{\mathbb{P}}^o(\Delta_i^{(L)} = 1) = \phi'(L) := 2[\overline{\mathbb{P}}^o(D' = \infty) - \phi(L)]^{-1}\phi(L)\,, \tag{1.17}$$

  such that the law of $(\tilde{\tau}_1^{(L)}, \tilde{X}_1^{(L)})$ is identical to the law of $(\bar{\tau}_1^{(L)}, \bar{X}_1^{(L)})$ under the measure $\overline{\mathbb{P}}^o[\,\cdot\,|D' = \infty]$,

- and another sequence $\{(Z_i^{(L)}, Y_i^{(L)})\}_{i\geq 1}$ such that

$$(\overline{\tau}_i^{(L)}, \overline{X}_i^{(L)}) = (1 - \Delta_i^{(L)})(\tilde{\tau}_i^{(L)}, \tilde{X}_i^{(L)}) + \Delta_i^{(L)}(Z_i^{(L)}, Y_i^{(L)})\,,$$

  and such that $\Delta_i^{(L)}$ is independent of $\{\tilde{\tau}_j^{(L)}\}_{j\leq i-1}, \{\tilde{X}_j^{(L)})\}_{j\leq i-1}, \{\Delta_j^{(L)}\}_{j\leq i-1}$ and of $(Z_i^{(L)}, Y_i^{(L)})$.

The joint law of the variables $\{(Z_i^{(L)}, Y_i^{(L)})\}_{i\geq 1}$ is complicated, but $|Y_i^{(L)}| \leq Z_i^{(L)}$ and $|\Delta_i^{(L)} Z_i^{(L)}| \leq \overline{\tau}_i^{(L)}$. In [2], we used the following integrability condition:

Assumption 1.18

($\mathcal{A}5$) *There exist an $\alpha > 2$ and $M = M(L)$ such that $\phi'(L)^{1/\alpha'} M(L)^{1/\alpha} \underset{L\to\infty}{\longrightarrow} 0$ (with $1/\alpha' = 1 - 1/\alpha$), and*

$$\overline{\mathbb{P}}^o\Big(\overline{\mathbb{E}}^o\big((\overline{\tau}_1^{(L)})^\alpha \mid D' = \infty, \mathcal{F}_0^L\big) > M \Big) = 0\,, \tag{1.19}$$

where $\mathcal{F}_0^L = \sigma(\omega(y,\cdot) : \ell\cdot y < -L)$.

We now recall the law of large numbers [2].

Theorem 1 *Assume either ($\mathcal{A}1, 2, 3$) and ($\mathcal{A}5$) for some $\alpha > 1$, or ($\mathcal{A}1, 2, 4$). Then, there exists a deterministic vector v with $v \cdot \ell > 0$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v\,, \quad \mathbb{P}^o - a.s..$$

Moreover, we give in Section 5 of [2], various non-nestling examples where ($\mathcal{A}5$) hold: In the course of Theorem 5.1 therein, we prove that, under condition ($\mathcal{A}3$) with sufficiently large δ, $M(L)$ grows at most exponentially. More precisely, for $\delta > \delta_1(\kappa, \alpha)$,

$$M(L) \leq e^{mL}\,, \quad L \geq 1\,, \tag{1.20}$$

with $m = m(\kappa, \alpha, \delta)$ finite. (The proof, given for $\alpha = 2$ in [2], extends to $\alpha > 2$.)

1.4 Main Result

Under strong mixing assumptions of the form of Assumption ($\mathcal{A}1'$), we can give a full invariance principle for the RWRE, and a law of large numbers, under integrability conditions slightly weaker than ($\mathcal{A}5$). Namely, set

Assumption 1.21

($\mathcal{A}5'$) *There exist an $\alpha > 2$ and $M = M(L)$ such that*

$$\overline{\mathbb{P}}^o\Big(\overline{\mathbb{E}}^o\big((\overline{\tau}_1^{(L)})^\alpha \mid D' = \infty, \mathcal{F}_0^L\big) > M \Big) = 0\,, \tag{1.22}$$

where $\mathcal{F}_0^L = \sigma(\omega(y,\cdot) : \ell\cdot y < -L)$.

We can now state out main result:

Theorem 2 (Annealed CLT, strong mixing) .
Assume ($\mathcal{A}1', 2, 3, 5'$). Then there exist a deterministic, non-degenerate covariance matrix R and a deterministic vector v such that under $\mathbb{P}^o$, with $S_n(t) := [X_{[nt]} - vt]/\sqrt{n}$, the path $S_n(t)$, taking values in the space of right continuous functions possessing left limits equipped with the supremum norm, converges weakly to a standard Brownian motion of covariance R.

Note that when strong mixing is available, Theorem 2 yields the law of large numbers under weaker integrability assumptions than those used in Theorem 1.

Remark: It is worthwhile to note that the statement of Theorem 2 and its proof carry over to the case where P is the marginal on $S^{\mathbb{Z}^d}$ of a strong mixing Gibbs Markov field on $(S \times S')^{\mathbb{Z}^d}$ with S' any compact Polish space. In the sake of aleviating notations, we do not pursue this remark further.

Our results for mixing environments satisfying only ($\mathcal{A}1$) are considerably weaker. With $M(L)$ from ($\mathcal{A}5$), $\phi'(L)$ from ($\mathcal{A}1$) and (1.17), we will assume the existence of sequences $L = L(n)$ and $k_n = k(L(n), n) \to_{n\to\infty} \infty$ such that

$$\frac{M(L)}{\kappa^{\frac{L\alpha}{2}} k_n^{\left(\frac{\alpha}{2}-1\right)}} \underset{n\to\infty}{\longrightarrow} 0, \tag{1.23}$$

and

$$M(L)^{\frac{1}{\alpha}} \phi'(L)^{\frac{1}{\alpha'}} \sqrt{\kappa^{-L} k_n} \underset{n\to\infty}{\longrightarrow} 0. \tag{1.24}$$

Theorem 3 (Annealed Gaussian behaviour, weak mixing) .
a) Assume ($\mathcal{A}1, 2, 3, 5$). Further, assume that sequences $L = L_n$ and k_n can be found that satisfy (1.23), (1.24), and the additional condition

$$\frac{n}{\overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)} \kappa^{-L}} \underset{n\to\infty}{\longrightarrow} 1. \tag{1.25}$$

Then, there exist a sequence of deterministic vectors $v(n)$, with $\lim_{n\to\infty} v(n) = v$, and a sequence of deterministic, positive definite, symmetric matrices R_n, defined in (3.17) below, such that with $R_n(w) = w^T R_n w$,

$$\lim_{n\to\infty} \left| \overline{\mathbb{P}}^o \left(\frac{X_n \cdot w - nv(n) \cdot w}{\sqrt{n}} \le x \right) - \overline{\mathbb{P}}^o \Big(\mathcal{N}(0, R_n(w)) \le x \Big) \right| = 0 \tag{1.26}$$

for all $x \in \mathbb{R}$ and all $w \in \mathbb{R}^d$. (According to the context, we denote by $\mathcal{N}(a, B)$ the Normal distribution of mean a and covariance matrix B, or a r.v. with this law.)
b) If (1.23) and (1.24) hold with $L \to \infty$, $k_n = k_L = e^{cL}$ for some constant $c > 0$, then one can

find sequences $L_n \in |\ell|_1\mathbb{N}^*$ *and* $k_n = k(L_n, n)$ *satisfying (1.23), (1.24) and (1.25).*
c) In the finite-dependence case (ie, $\phi'(L) = 0$ *for* $L \geq L_0$*), we can keep* $L = L_0$ *fixed. In this case,* $v(n) = v$ *and* $R_n = R$*, a positive definite matrix independent of* n*, and the statement is the standard central limit theorem:*

$$R^{-1/2}(X_n - nv) \longrightarrow \mathcal{N}(0, \mathrm{Id}) \qquad \text{in law.}$$

Remark: In view of (1.20), we see that conditions $(\mathcal{A}1, 2, 3, 5)$ with $\delta > \delta_1$ and $\phi(r) \leq e^{-\gamma r}$ with large enough γ ensures that part b) of Theorem 3 applies. Hence, Theorem 3 applies to both non-nestling and nestling walks. On the other hand, we do not control in any way the convergence or non-degeneracy of the sequence of covariances R_n, and cannot rule out sub or super diffusive behaviour in the generality of assumption $\mathcal{A}1$.

2 Proof of Theorem 2

The key to the proof in the strong mixing case is to consider the sequence of truncated cones of the environment produced by the regeneration times. To formalize this, define the space $\mathcal{T}$ of truncated cone environments and paths as

$$\mathcal{T} = \bigcup_{M = y\cdot\ell > 0, y \in \mathbb{Z}^d} \{M\} \times \mathcal{P}_M \times S^{C(0,\ell,\zeta,M)},$$

where the space of finite paths in the truncated cone $C(0, \ell, \zeta, M)$ (cf (1.4)) is defined as

$$\mathcal{P}_M = \{\underline{x} = (x_1, \ldots, x_k) \in C(0, \ell, \zeta, M)^{\mathbb{N}_*} : x_0 = 0, |x_{i+1} - x_i| = 1\}.$$

Set $\overline{\mathcal{T}} = \mathcal{T} \cup \{\mathtt{s}\}$, where $\mathtt{s}$ is an extra stop symbol. We set $\mathcal{W} = \overline{\mathcal{T}}^{\mathbb{N}}$ as the space of infinite words consisting of finite truncated cones environments and finite cone based paths, with the restriction that if $w_i = \mathtt{s}$ then $w_j = \mathtt{s}$ for all $j > i$. Note that finite words of length k can be naturally viewed as elements of $\mathcal{W}$ by setting $w_i = \mathtt{s}$ for all $i > k$. $\mathcal{W}$ inherits naturally a Borel structure that makes it into a measure space. We further define on $\mathcal{W}$ a lexicographic distance as

$$d(w, w') = 2^{-\min\{i : w_i \neq w'_i\}}.$$

Next, we fix L and note that the sequence $t_k = \tau_{k+1}^{(L)} - \tau_k^{(L)}$, $k \geq 1$, and the RWRE path X_n define an element $\underline{r} = (r_1, r_2, \ldots) \in \mathcal{W}$ via

$$r_k = \left(t_k - L, \{X_{j+\tau_k^{(L)}} - X_{\tau_k^{(L)}}\}_{j=1,\ldots,t_k-L}, \{\omega_y\}_{y \in C(X_{\tau_k^{(L)}}, \ell, \zeta, (X_{\tau_{k+1}^{(L)}} - X_{\tau_k^{(L)}})\cdot\ell - \ell_L)}\right), \tag{2.1}$$

where $\ell_L = L|\ell|^2/|\ell|_1$ is an integer by our restriction on the allowed L and ℓ. However, this will not be particularly useful to us as we think of $\mathcal{W}$ as a sequence of $\mathcal{T}$ valued symbols extending

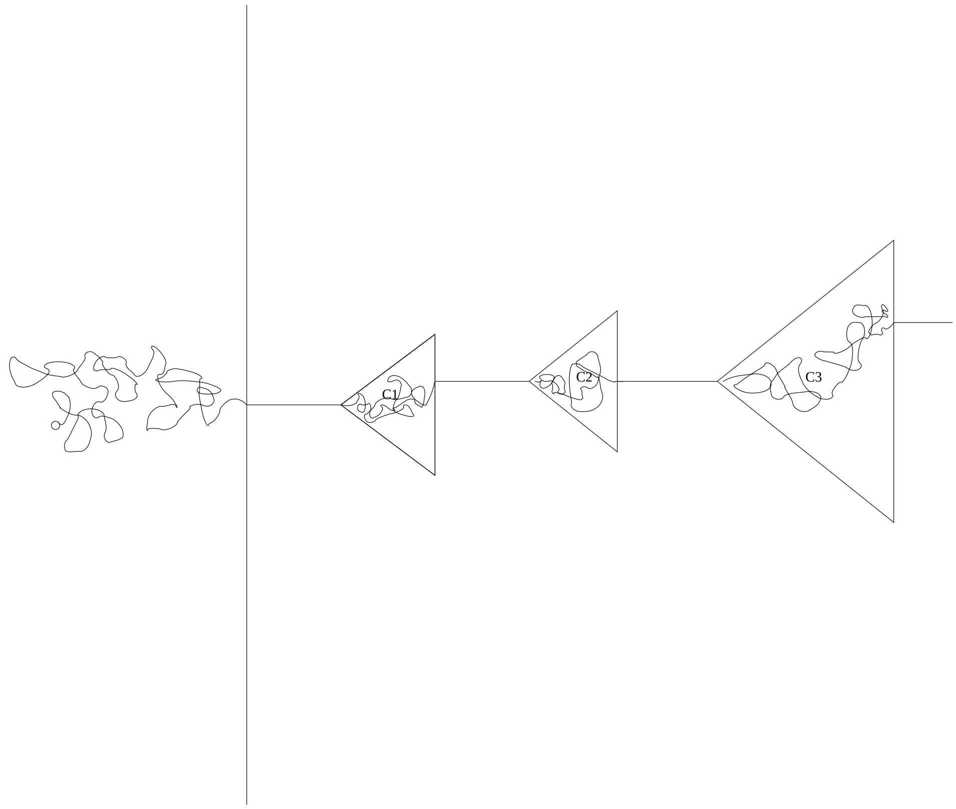

Figure 1: The sequence $\underline{r} = (r_1, r_2, \ldots)$, with $\ell = (1, 0, \ldots, 0)$. The hyperplane (to the right of the origin) is determined by the first regeneration location $X_{\tau_1^{(L)}}$, and $\mathcal{H}_1$ is determined by the path up to that location and the environment to the left of this first hyperplane. Shown are the cones C_1, C_2, C_3 as in the proof of Lemma 2.2, the random walk path inside the cones, and the directed paths between the cones (of length L) determined by the sequence ε.

backward in time, and it will be convenient to think of $\underline{r}$ as defining a sequence of words $w^{(k)} = (r_k, r_{k-1}, \ldots, r_1, \mathtt{s}, \ldots) \in \mathcal{T}^k$. Further, recall from [2] the sigma-fields

$$\mathcal{H}_1 = \sigma\left(\tau_1^{(L)}, X_0, \varepsilon_0, X_1, \cdots, \varepsilon_{\tau_1^{(L)}-1}, X_{\tau_1^{(L)}}, \{\omega(y,\cdot); \ell \cdot y < \ell \cdot X_{\tau_1^{(L)}} - L|\ell|^2/|\ell|_1\}\right),$$

$$\mathcal{H}_k = \sigma\Big(\tau_1^{(L)} \ldots \tau_k^{(L)}, \quad X_0, \varepsilon_0, X_1, \cdots, \varepsilon_{\tau_k^{(L)}-1}, X_{\tau_k^{(L)}}, \quad \{\omega(y,\cdot); \ell \cdot y < \ell \cdot X_{\tau_k^{(L)}} - L|\ell|^2/|\ell|_1\}\Big),$$

and set

$$\mathcal{U} = \{(m, y_1, \ldots, y_m, \omega'); m \geq 1, y_i \in \mathbb{Z}^d, |y_{i+1} - y_i| = 1, y_m \cdot \ell > y_i \cdot \ell, \ \forall i < m, \\ \omega' \in S^{\mathbb{Z}^d \setminus \{z : z \cdot \ell > y_m \cdot \ell\}}\}.$$

Then, $\overline{\mathbb{P}}^o$ induces a probability distribution $\mathbb{Q}^o$ on $\mathcal{U}$ such that, for $B \in \mathcal{H}_1$, $B = \cup_{t\in\mathbb{N}_*, z\in\mathbb{Z}^d} B_{t,z}$ with $B_{t,z} = B \cap \{\tau_1^{(L)} - L = t, X_{\tau_1^{(L)}-L} = z\}$, one has

$$\mathbb{Q}^o(B_{t,z}) = \overline{\mathbb{P}}^o((\tau_1^{(L)} - L, X_1, \ldots, X_t, \{\omega_y\}_{y\cdot\ell\leq z\cdot\ell}) \in B_{t,z}),$$

and the law $\overline{\mathbb{P}}^o(\cdot|\mathcal{H}_1)$ induces on the sequence $\underline{r}$ a probability distribution such that the (random) kernels $h_{u,i}(\cdot|w_{i-1}, \ldots, w_2, w_1)$, $u \in \mathcal{U}$ are well defined by the following: for each integer k, each measurable $A \subset \mathcal{T}^k$, and each measurable $B \in \mathcal{H}_1$,

$$\overline{\mathbb{E}}^o\left[\mathbf{1}_B \overline{\mathbb{P}}^o((r_1, \ldots, r_k) \in A|\mathcal{H}_1)\right] = \int_B \mathbb{Q}^o(du) \int_{\mathcal{T}} \cdots \int_{\mathcal{T}} \mathbf{1}_A \prod_{i=1}^k h_{u,i}(du_i|u_{i-1}, \ldots, u_1).$$

(To define the kernels $h_{u,i}$, simply note that that $\overline{\mathbb{P}}^o(r_k \in A|\mathcal{H}_k)$ defines a measurable function on $\mathcal{U} \times \mathcal{T}^{k-1}$, which is exactly $h_{u,i}(A|u_{i-1}, \ldots, u_1)$).

The following lemma is crucial to our approach.

Lemma 2.2 *Let $i' \geq i$, $\underline{u}^{(i)} = (u_i, \ldots, u_1)$ and $\underline{u'}^{(i')} = (u'_{i'}, \ldots, u'_1)$ be such that $u_{i-j} = u'_{i'-j}$ for $j = 0, \ldots i_0$. Then,*

$$\sup_{u,u'\in\mathcal{U}} \|h_{u,i+1}(\cdot|\underline{u}^{(i)}) - h_{u',i'+1}(\cdot|\underline{u'}^{(i')})\|_{\mathrm{Var}} \leq \phi(i_0 L). \tag{2.3}$$

Proof of Lemma 2.2: The proof is a modification of the argument in [2, Lemma 2.13], using the strong mixing assumption. Especially the case $i = i' = i_0 = 1$, is a slight variation of the proof given in [2, Lemma 2.13].

For $\underline{u}^{(i)}, \underline{u'}^{(i')}$ from $\underline{u}, \underline{u'}$ infinite sequences in $\mathcal{T}$, we observe that the maximum over $i, i' \geq i_0$ of the left-hand side of (2.3) is achieved with $i = i' = i_0$, therefore we need to consider only the latter case.

We first note that the values $u, u_1, \ldots, u_i$ determine a sequence of points $\bar{x}_i \in \mathbb{Z}^d$ and times $\bar{t}_i \in \mathbb{N}_*$ that encode the regeneration locations and times. More precisely, if $u = (m, y_1, \ldots, y_m, \omega_{H_u})$ for the appropriate half space $H_u = \{x; x\cdot\ell \leq y_m \cdot \ell\}$, and if $u_i = (m_i, x_1^{(i)}, \ldots, x_{k_i}^{(i)}, \omega_{C_i})$ for some truncated cone C_i, we let p denote the projection on $\mathcal{T}$ given by $p(u_i) = (m_i, x_1^{(i)}, \ldots, x_{k_i}^{(i)})$. Then, the regeneration locations and times are equal to

$$\bar{x}_0 = y_m + L\ell/|\ell|_1, \quad \bar{x}_i = \bar{x}_{i-1} + [x_{k_i}^{(i)} + L\ell/|\ell|_1], \quad \bar{t}_i = m + L + \sum_{j=1}^i [k_j + L].$$

In fact, from $(p(u), p(u_1), \ldots p(u_i))$, the whole path on the time interval $[0, \bar{t}_i]$ can be reconstructed: we denote by $\tilde{x} = \tilde{x}[(p(u), p(u_1), \ldots p(u_i))]$ this finite path – in particular, $\tilde{x}(\bar{t}_k) = \bar{x}_k$.

Let A be a measurable subset of $\mathcal{T}$, and write for short $\mathbf{1}_A = \mathbf{1}_{r_0 \in A}$, where r_0 is defined by (2.1) for $k=0$, with $\tau_0^{(L)} = 0$.

Let also $F \geq 0$ be a $\mathcal{H}_1$-measurable bounded random variable [resp., $G \geq 0$ bounded measurable on $\sigma(r_1, \ldots r_i)$]. Then for all $p_0 \in p(\mathcal{T}), \underline{p}^{(i)} \in p(\mathcal{T})^i$, there exist random variables F_{p_0} [resp., $G_{\underline{p}^{(i)}}$], measurable with respect to $\sigma(\{\omega(y,\cdot); y \cdot \ell < y_m \cdot \ell\}, \{\varepsilon_k; 1 \leq k \leq m\})$ [resp., $\sigma(\{\omega(y,\cdot); y \in U\}$] such that, on the event $\{p(r_0) = p_0\}$ it holds $F = F_{p_0}$ [resp., on the event $\{p(r_k) = p_k, 1 \leq k \leq i\}$ it holds $G = G_{\underline{p}^{(i)}}$]. Throughout, we use the notation $U = \cup_{j=1}^{i}(C_i + \bar{x}_{i-1})$, and we define the events $C(p_0) = \{X_k = \tilde{x}_k; 0 \leq k \leq \bar{t}_0\}$, $B(\underline{p}^{(i)}) = \{X_{k+\bar{t}_0} - X_{\bar{t}_0} = \tilde{x}_{k+\bar{t}_0} - \tilde{x}_{\bar{t}_0}; 0 \leq k \leq \bar{t}_i - \bar{t}_0\}$ with $\tilde{x} = \tilde{x}[(p(u), p(u_1), \ldots p(u_i))]$. By the Markov property,

$$
\begin{aligned}
&\overline{\mathbb{E}}^o(FG\, \mathbf{1}_A \circ \theta_{\tau_{i+1}^{(L)}}) = \overline{\mathbb{E}}^o(FG\, \mathbf{1}_{A \bigcap \{D'=\infty\}} \circ \theta_{\tau_{i+1}^{(L)}}) = \\
&= \sum_{(p_0, \underline{p}^{(i)}) \in p(\mathcal{T})^{i+1}} E_{P\otimes Q} \overline{E}^o_{\omega,\varepsilon}(F_{p_0} \mathbf{1}_{C(p_0)} G_{\underline{p}^{(i)}} \mathbf{1}_{B(\underline{p}^{(i)})}\, \mathbf{1}_{A \bigcap \{D'=\infty\}} \circ \theta_{\bar{t}_i} \\
&= \sum_{p_0, \underline{p}^{(i)}} E_{P\otimes Q} \left[\overline{E}^o_{\omega,\varepsilon}(F_{p_0} \mathbf{1}_{C(p_0)} G_{\underline{p}^{(i)}} \mathbf{1}_{B(\underline{p}^{(i)})} \times \overline{P}^{\bar{x}_i}_{\omega, \theta_{\bar{t}_i}\varepsilon}(A \bigcap \{D'=\infty\}) \right] \\
&= \sum_{p_0, \underline{p}^{(i)}} E_{P\otimes Q} \left[E_{P\otimes Q} \left(\overline{E}^o_{\omega,\varepsilon}(F_{p_0} \mathbf{1}_{C(p_0)} G_{\underline{p}^{(i)}} \mathbf{1}_{B(\underline{p}^{(i)})} \times \overline{P}^{\bar{x}_i}_{\omega, \theta_{\bar{t}_i}\varepsilon}(A \bigcap \{D'=\infty\}) \,\Big|\, \omega_z, z \in U \right) \right] \\
&= \sum_{p_0, \underline{p}^{(i)}} E_{P\otimes Q} \left[\overline{G}_{p_0, \underline{p}^{(i)}} \times E_{P\otimes Q} \left(F_{p_0} \overline{P}^0_{\omega,\varepsilon}(C(p_0)) \times \overline{P}^{\bar{x}_i}_{\omega, \theta_{\bar{t}_i}\varepsilon}(A \bigcap \{D'=\infty\}) \,\Big|\, \omega_z, z \in U \right) \right]
\end{aligned}
$$

where we have set

$$
\overline{G}_{p_0, \underline{p}^{(i)}} = G_{\underline{p}^{(i)}} \overline{P}^0_{\omega,\varepsilon} \Big(B(\underline{p}^{(i)}) | X_l, l \leq \bar{t}_0, X_{\bar{t}_0} = \bar{x}_0 \Big) \,,
$$

which is $\sigma(\omega_z, z \in U)$-measurable. Define $h_{\emptyset, i+1}(\cdot | \underline{u}^{(i)})$ the conditional law of r_{i+1} given $r_1, \ldots r_i$, and define also ρ_A by

$$
\rho_A = \sum_{p_0, \underline{p}^{(i)}} E_{P\otimes Q} \left[\overline{G}_{p_0, \underline{p}^{(i)}} \times \mathrm{Cov}_{P(\cdot|\omega_z, z\in U)} \left\{ \overline{P}^{\bar{x}_i}_{\omega, \theta_{\bar{t}_i}\varepsilon}(A \bigcap \{D'=\infty\}); F_{p_0} \overline{P}^o_{\omega,\varepsilon}(C(p_0) \right\} \right] \tag{2.4}
$$

allowing to write

$$
\begin{aligned}
&\overline{\mathbb{E}}^o(FG\, \mathbf{1}_A \circ \theta_{\tau_{i+1}^{(L)}}) = \\
&\quad = \rho_A + \sum_{p_0, \underline{p}^{(i)}} E_{P\otimes Q} \Big[\overline{G}_{p_0, \underline{p}^{(i)}} \times \\
&\qquad E_{P\otimes Q} \left(\overline{P}^{\bar{x}_i}_{\omega, \theta_{\bar{t}_i}\varepsilon}(A \bigcap \{D'=\infty\}) \,\Big|\, \omega_z, z \in U \right) \times E_{P\otimes Q} \left(F_{p_0} \overline{P}^o_{\omega,\varepsilon}(C(p_0)) \,\Big|\, \omega_z, z \in U \right) \Big] \\
&\quad = \rho_A + \sum_{p_0, \underline{p}^{(i)}} E_{P\otimes Q} \left[h_{\emptyset, i+1}(A | \underline{u}^{(i)}) \overline{G}_{p_0, \underline{p}^{(i)}} E_{P\otimes Q} \left(F_{p_0} \overline{P}^o_{\omega,\varepsilon}(C(p_0)) \,\Big|\, \omega_z, z \in U \right) \right] \qquad (2.5)
\end{aligned}
$$

where we have used that

$$E_{P\otimes Q}\left(\overline{P}^{\bar{x}_i}_{\omega,\theta_{\bar{t}_i}\varepsilon}(A\bigcap\{D'=\infty\})\,\Big|\omega_z,z\in U\right)=h_{\emptyset,i+1}(A|\underline{u}^{(i)})$$

holds on the set $B(\underline{p}^{(i)})\bigcap C(p_0)$, by definition of $h_{\emptyset,i+1}$ and since the sequence ε is i.i.d. Observe at this point that, by definition of $F_{p_0},\overline{G}_{p_0}$,

$$\begin{aligned}\overline{\mathbb{E}}^o(FGh_{\emptyset,i+1}(A|r^{(i)})) &= \sum_{p_0,\underline{p}^{(i)}}\overline{\mathbb{E}}^o(h_{\emptyset,i+1}(A|\underline{u}^{(i)})G_{\underline{p}^{(i)}}\mathbf{1}_{B(\underline{p}^{(i)})}F_{p_0}\mathbf{1}_{C(p_0)})\\ &= \sum_{p_0,\underline{p}^{(i)}}\overline{\mathbb{E}}^o\left[h_{\emptyset,i+1}(A|\underline{u}^{(i)})\overline{G}_{p_0,\underline{p}^{(i)}}E_{P\otimes Q}\left(F_{p_0}\overline{P}^o_{\omega,\varepsilon}(C(p_0))\,\Big|\omega_z,z\in U\right)\right] \qquad (2.6)\end{aligned}$$

Thus, (2.5) reads

$$\overline{\mathbb{E}}^o(FG\,\mathbf{1}_A\circ\theta_{\tau^{(L)}_{i+1}})=\rho_A+\overline{\mathbb{E}}^o(FGh_{\emptyset,i+1}(A|r^{(i)})) \qquad (2.7)$$

The crucial point to observe is that since g is measurable with respect to $\sigma(\omega_x,x\in C_{i+1}+\bar{x}_i)$, the strong mixing property (1.8) implies that, a.s.,

$$|E(g|\omega_x,x\in H_u\cup U)-E(g|\omega_x,x\in U)|\le\phi(iL)||g||_\infty\,,$$

with $\phi(r)=C'e^{-\gamma r/2}$. Hence, since f is measurable with respect to $\sigma(\omega_x,x\in H_u)$, this results in

$$|E(fg|\omega_x,x\in U))-E(f|\omega_x,x\in U)E(g|\omega_x,x\in U)|\le\phi(iL)E(|f||\omega_x,x\in U)||g||_\infty\,,$$

replacing [2, Equation (1.5)]. Hence, from (2.4),(2.6)

$$|\rho_A|\quad\le\quad\phi(iL)\overline{\mathbb{E}}^o(FG)\,.$$

Finally one obtains from (2.7)

$$|\overline{\mathbb{E}}^o(FG\,\mathbf{1}_A\circ\theta_{\tau^{(L)}_{i+1}})-\overline{\mathbb{E}}^o(FGh_{\emptyset,i+1}(A|r^{(i)}))|\le\phi(iL)\overline{\mathbb{E}}^o(FG)\,,$$

which is enough to prove the lemma. □

Lemma 2.2 allows us to have, with $M_1(\mathcal{T})$ denoting the space of probability measures on $\mathcal{T}$:

Lemma 2.8 *There exists a measurable kernel $h:\mathcal{W}\mapsto M_1(\mathcal{T})$ such that*

$$\sup_{k\ge i,u\in\mathcal{U},\underline{w}\in\mathcal{T}^{k-1},w'\in\mathcal{W}:d(w,w')<2^{-i}}\|h_{u,k}(\cdot|\underline{w})-h(\cdot|w')\|_{\mathrm{Var}}\;<\phi(iL)\,, \qquad (2.9)$$

and

$$\sup_{w\in\mathcal{W},w'\in\mathcal{W}:d(w,w')<2^{-k}}\|h(\cdot|w)-h(\cdot|w')\|_{\mathrm{Var}}\;<2\phi(kL)\,. \qquad (2.10)$$

Proof Fix $u \in \mathcal{U}$ and $w = (w_1, w_2, \ldots) \in \mathcal{W}$, setting $w^{(k)} = (w_1, \ldots w_k)$. Note that by Lemma 2.2, the sequence $(h_{u,k}(\cdot|w^{(k-1)}), k \geq 1)$ forms a Cauchy sequence with respect to the variation distance between elements of $M_1(\mathcal{T})$, with

$$\sup_{u,u',w\in\mathcal{W}} \|h_{u,k}(\cdot|w^{(k-1)}) - h_{u',k'}(\cdot|w^{(k'-1)})\|_{\mathrm{Var}} \leq \phi((k' \wedge k)L)\,.$$

The existence of a limit $h_u(\cdot|w)$ follows from completeness of $M_1(\mathcal{T})$, together with the estimate

$$\sup_{u,u',w\in\mathcal{W}} \|h_{u',k}(\cdot|w^{(k-1)}) - h_u(\cdot|w)\|_{\mathrm{Var}} \leq \sum_{k'\geq k} \phi(kL)\,.$$

One deduces that h_u in fact does not depend on u, and the estimate in (2.10). $\square$

We next note that the kernel h and initial condition $w \in \mathcal{W}$ determine a Markov chain $\{w(n)\}_{n\geq 0}$ with state space $\mathcal{W}$, with law denoted $P_w(\cdot)$. Indeed, with $y \in \mathcal{T}$, $w \in \mathcal{W}$, define $yw \in \mathcal{W}$ by setting $(yw)_1 = y$ and $(yw)_i = w_{i-1}$ for $i \geq 2$. Then, with $w(n) \in \mathcal{W}$, let $y(n+1)$ be distributed according to $h(\cdot|w(n))$, and set $w(n+1) = (y(n+1)w(n))$. Further, by Lemma 2.8, the Markov chain satisfies conditions FLS($\mathcal{T}$,1) and M(1) of [5, Pages 47,51]. Hence, by [5, Theorem 2.27], it is uniformly ergodic and possesses a unique invariant distribution. Further, for $y \in \mathcal{T}$ with $y = (m, \underline{x}, \underline{s}) \in \mathcal{T}$ and $\underline{x} = (x_1, \ldots, x_m)$, define $f(y) = x_m$. Fatou's lemma and condition ($\mathcal{A}$5') then imply the integrability condition

$$\sup_w \int |f(y)|^\alpha h(dy|w) < \infty\,. \tag{2.11}$$

Further, setting $g(y) = m$, the law of large numbers ([5, Proposition 4.1.1 and Theorem 4.1.2]) and another application of ($\mathcal{A}$5') imply that

$$\frac{1}{n}\sum_{i=1}^n g(w(i)_1) \to_{n\to\infty} C_1\,, \quad \frac{1}{n}\sum_{i=1}^n f(w(i)_1) \to_{n\to\infty} C_2\,, \tag{2.12}$$

almost surely, with C_1, C_2 being deterministic and equal to the expectation of $g(w_1), f(w_1)$, respectively, under the unique invariant measure mentioned above.

Next, by [5, Theorem 4.1.5] and (2.11), and the ϕ mixing of the sequence $f(w(i)_1)$ ensured by [5, Theorem 2.1.5], the invariance principle holds, under P_w, for $Z_n(t)$, where

$$Z_n(t) := \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} [f(w(i)_1) - C_2 g(w(i)_1)/C_1]\,,$$

with variance that does not depend on the initial condition w.

It thus remains to transfer the statement of the invariance principle from the Markov chain $\{w(n)\}_{n\geq 0}$ to the original sequence S_n. Toward this end, define

$$\tilde{S}_n(k) = \frac{X_{\tau_k^{(L)}} - \frac{C_2 \tau_k^{(L)}}{C_1}}{\sqrt{n}}\,,$$

and recall that by [2, Lemma 3.13], there exist deterministic positive sequences β_L and $\eta_L \to 0$ such that

$$\limsup_{n\to\infty} n^{-1}|\tau_n^{(L)} - \beta_L \kappa^{-L} n| < \eta_L, \quad \overline{\mathbb{P}}^o - a.s., \quad \liminf_{L\to\infty} \beta_L \geq t_{\rm av} > 0\,, \tag{2.13}$$

see [2, (3.6)] for the last fact. We assume throughout that L is chosen large enough such that both $\phi(L) < 1$ and $\eta_L < t_{\rm av}/2$.

Next, fix $\varepsilon \in (0,1)$ and $w \in \mathcal{W}$. Due to Lemma 2.8, and the fact that $\sum_k \phi(kL) < \infty$, one may find a sequence $k_0(\varepsilon) < \infty$ (with $k_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$) such that it is possible to construct a probability space with probability measure denoted $\tilde{\mathbb{P}}$ (both depending on ε, w) on which there exist:

- a sequence $(r_k)_k$ distributed according to $\overline{\mathbb{P}}^o(\underline{r} \in \cdot|\mathcal{H}_1)$, with $\underline{r}$ from (2.1),
- a sequence $w(n)$ distributed according to P_w,

such that

$$\tilde{\mathbb{P}}\left(\exists k \geq \delta(\varepsilon) : r_k \neq w(k)_1\right) \leq \varepsilon\,. \tag{2.14}$$

Indeed, in view of Lemma 2.8, we can recursively couple $(r_k)_k$ and $(w(k))_k$ so that

$$\tilde{\mathbb{P}}(r_{i+1} = w(i+1)_1 | r_1, \ldots r_i, w(1)_1, \ldots w(i)_1) \geq 1 - \phi(kL) \text{ on } \{r_l = w(l)_1, i \geq l \geq i-k+1\}\,.$$

Then, (2.14) follows easily form $\sum_k \phi(kL) < \infty$.

Further, note that $\overline{\mathbb{P}}^o$-a.s., $\forall \delta > 0$,

$$\begin{aligned}
&\overline{\mathbb{P}}^o\left(\sup_{k=1}^{n} \sup_{\tau_k^{(L)} \leq t \leq \tau_{k+1}^{(L)}} \left[\left|X_t - X_{\tau_k^{(L)}}\right|_1 + \left|t - \tau_k^{(L)}\right|\right] > 2\delta\sqrt{n}\,\Big|\mathcal{H}_1\right) \\
\leq\;& n \sup_{k=1}^{n} \overline{\mathbb{P}}^o\left(\left|\tau_{k+1}^{(L)} - \tau_k^{(L)}\right| \geq \delta\sqrt{n}\,|\mathcal{H}_k\right) \\
\leq\;& n \operatorname{esssup} \overline{\mathbb{P}}^o\left(\tau_1^{(L)} \geq \delta\sqrt{n}\,|D' = \infty, \mathcal{F}_0^L\right) \\
\leq\;& \frac{n}{(\delta\sqrt{n})^\alpha} \operatorname{esssup} \overline{\mathbb{E}}^o\left((\tau_1^{(L)})^\alpha\,|D' = \infty, \mathcal{F}_0^L\right) \\
\leq\;& \frac{nM(L)\kappa^{-\alpha L}}{(\delta\sqrt{n})^\alpha} \to_{n\to\infty} 0\,.
\end{aligned} \tag{2.15}$$

For any fixed T deterministic, set $J_T = 2(T+1)/t_{\rm av}\kappa^{-L}$. Note that, by construction and in view of (2.14),

$$\tilde{\mathbb{P}}\left(\sup_{\delta(\varepsilon) \leq k \leq nJ_T} \left|\tilde{S}_n(k) - \tilde{S}_n(\delta(\varepsilon)) - Z_n\left(\frac{k}{n}\right) + Z_n\left(\frac{\delta(\varepsilon)}{n}\right)\right|_1 > 0\right) \leq \varepsilon\,. \tag{2.16}$$

Further,

$$\overline{\mathbb{P}}^o(\sup_{t\le\tau_1^{(L)}} |X_t|_1 > \delta\sqrt{n}) \le \overline{\mathbb{P}}^o(|\tau_1^{(L)}| > \delta\sqrt{n}) \to_{n\to\infty} 0\,. \tag{2.17}$$

It follows from (2.15), (2.16) and (2.17), by taking first $n\to\infty$ and then $\varepsilon\to 0$, that the invariance principle for Z_n carries over to an invariance principle, under the measure $\overline{\mathbb{P}}^o$, for $\tilde{S}_n([tn])$, on the interval $0\le t\le J_T$, with the same non-degenerate limit covariance. On the other hand, by the law of large numbers (2.12) and (2.15),

$$\limsup_{n\to\infty}\overline{\mathbb{P}}^o\left(\sup_{k\le nJ_T}\left|\frac{\tau_k^{(L)}}{n}-C_2\frac{k}{n}\right|>\delta\right)\le\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\tilde{\mathbb{P}}\left(\sup_{k\le nJ_T}\left|\frac{\tau_k^{(L)}}{n}-C_2\frac{k}{n}\right|>\delta\right)=0\,, \tag{2.18}$$

while, by (2.14),

$$\limsup_{n\to\infty}\overline{\mathbb{P}}^o(\tau_{nJ_T}^{(L)}<Tn)\le\limsup_{\varepsilon\to 0}\limsup_{n\to\infty}\tilde{\mathbb{P}}(\tau_{nJ_T}^{(L)}<Tn)=0\,. \tag{2.19}$$

Hence, by the stability of the invariance CLT by random time changes [1, Theorem 14.4] together with (2.15), one concludes the invariance principle for $S_n(t)-C_2t/C_1$. $\square$

3 Proof of Theorem 3

Throughout this section, we assume without further mentioning it (A1,2,3,5).

Fix a direction w. The following preliminary lemma is easily proved

Lemma 3.1 *Assume (1.23). Then, with $\tilde{T}_n^{(L)}=\sum_{i=1}^n\tilde{\tau}_i^{(L)}$,*

$$\frac{\tilde{T}_{k_n}^{(L)}}{\overline{\mathbb{E}}^o\tilde{T}_{k_n}^{(L)}}\longrightarrow 1 \text{ in probability} \tag{3.2}$$

Proof: Recall that

$$\overline{\mathbb{E}}^o\tilde{\tau}_1^{(L)}\ge\frac{t_{av}}{2}>0\,,\quad\text{for all } L \text{ large} \tag{3.3}$$

by [2, (3.6)]. Now $\tilde{T}_{k_n}^{(L)}=\sum_{j=1}^{k_n}\tilde{\tau}_j^{(L)}$, and the $\tilde{\tau}_j^{(L)}$ are i.i.d. by construction. Further, by A5,

$$\overline{\mathbb{E}}^o\left(\tilde{\tau}_1^{(L)}\right)^2\le\overline{\mathbb{E}}^o\left([\tilde{\tau}_1^{(L)\alpha}]\right)^{\frac{2}{\alpha}}\le M(L)^{\frac{2}{\alpha}}$$

and hence, by (1.23),

$$\frac{\mathrm{Var}\ \tilde{T}_{k_n}^{(L)}}{(\overline{\mathbb{E}}^o\tilde{T}_{k_n}^{(L)})^2}\le\frac{k_nM(L)^{\frac{2}{\alpha}}}{k_n^2\left(\frac{t_{av}}{2}\right)^2}\underset{L\to\infty}{\longrightarrow}0.$$

□

Set

$$\tilde{S}_m^{(L)} = \sum_{i=1}^{m} \tilde{X}_i^{(L)} \quad , \qquad v_L = \frac{\overline{\mathbb{E}}^o \tilde{X}_1^{(L)}}{\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)}} \, ,$$

and

$$r_L = r_L(w) = \mathrm{Var}\ \left(w \cdot [\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L] \right) \, .$$

We recall from [2] that $v_L \to v$ as $L \to \infty$. For fixed $\delta > 0$, define

$$\tilde{Z}_\delta^{(L)}(w) = \tilde{Z}_\delta^{(L)}(w)(n) := \frac{w \cdot \left(\tilde{S}_{k_n}^{(L)} - \tilde{T}_{k_n}^{(L)} v_L \right)}{\sqrt{k_n} \left(\sqrt{r_L \vee \delta \kappa^L} \right)} \, .$$

The following lemma is the heart of our argument. Let denote by $\mathcal{L}(Z)$ the law of a random variable Z, and by ρ the Prohorov distance between probability measures.

Lemma 3.4 *Assume k_n satisfies (1.23), and set*

$$A_\delta(w) := \{ n : r_L > \delta \kappa^L \} \, .$$

Then,

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \rho \left(\mathcal{L} \left(\tilde{Z}_\delta^{(L)}(w) \right), \mathcal{N} \left(0, \left(\mathbf{1}_{A_\delta(w)}(n) + \mathbf{1}_{A_\delta^c(w)}(n) \frac{r_L}{\delta \kappa^L} \right) \right) \right) = 0 \, . \tag{3.5}$$

Proof of Lemma 3.4
Assume the statement does not hold true, that is that for some $\varepsilon_1 > 0$ the left hand side of (3.5) is larger than ε_1. Then, one may find $\delta > 0$ arbitrarily small and a sequence $n_k = n_k(\delta, \varepsilon_1)$ such that (we write $L = L_{n_k}$)

$$\rho \left(\mathcal{L} \left(\tilde{Z}_\delta^{(L)}(w)(n_k) \right), \left(\mathbf{1}_{A_\delta(w)}(n_k) + \mathbf{1}_{A_\delta^c(w)}(n_k) \sqrt{\frac{r_L}{\delta \kappa^L}} \right) \mathcal{N}(0,1) \right) > \frac{\varepsilon_1}{2} \, . \tag{3.6}$$

Then, fixing $\delta_1 < \delta$, either one of the following occurs:

a) There exists a further subsequence, still denoted by n_k, such that both (3.6) and $n_k \in A_\delta(w)$.

b) There exists a further subsequence, still denoted by n_k, such that both (3.6) and $n_k \in A_\delta(w)^c \cap A_{\delta_1}(w)$.

c) There exists a further subsequence, still denoted by n_k, such that both (3.6) and $n_k \in A_{\delta_1}(w)^c$.

Treating first case a), one applies the Lindeberg-Feller theorem (see e.g. [4, pg, 116, Theorem 4.5]). Indeed, one has on A_δ that $\tilde{Z}_\delta^{(L)}(w) = \tilde{Z}_0^{(L)}(w)$ and

$$\tilde{Z}_\delta^{(L)}(w) = \sum_{i=1}^{k_n} \frac{w \cdot (\tilde{X}_i^{(L)} - \tilde{\tau}_i^{(L)} v_L)}{\sqrt{k_n r_L}} \quad =: \sum_{i=1}^{k_n} Y_{i,L}$$

and $\sum_{i=1}^{k_n} \overline{\mathbb{E}}^o Y_{i,L}^2 = 1$.

Next, we see that on A_δ, using Hölder's and then Chebycheff's inequalities in the first and second inequalities and A5 in the third,

$$\begin{aligned}
\sum_{i=1}^{k_n} \overline{\mathbb{E}}^o \left(Y_{i,L}^2 \cdot \mathbf{1}_{|Y_i,L|>\varepsilon} \right) &= \overline{\mathbb{E}}^o \left(\frac{[w \cdot (\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L)]^2}{r_L} \mathbf{1}_{\frac{|w \cdot (\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L)|}{\sqrt{r_L}} \geq \varepsilon \sqrt{k_n}} \right) \\
&\leq \frac{1}{(\varepsilon \sqrt{k_n})^{\alpha-2}} \overline{\mathbb{E}}^o \left(\left[\frac{w \cdot (\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L)}{r_L^{1/2}} \right]^\alpha \right) \\
&\leq \frac{1}{(\varepsilon \sqrt{k_n})^{\alpha-2} (r_L)^{\alpha/2}} \overline{\mathbb{E}}^o \left(\tilde{\tau}_1^{(L)\alpha} \right) \\
&\leq \frac{M(L)}{\delta^{\frac{\alpha}{2}} \varepsilon^{\alpha-2} \kappa^{\frac{L\alpha}{2}} k_n^{\left(\frac{\alpha}{2}-1\right)}}
\end{aligned} \tag{3.7}$$

Using (1.23), one sees that the RHS in (3.7) converges to 0 with $n \to \infty$. This is enough in order to apply the Lindeberg-Feller theorem and conclude that for sequences $\{n_k\}$ in A_δ, $\tilde{Z}_\delta^{(L)}(w)$ converges in distribution to a standard Gaussian, contradicting (3.6).

Considering next case b), the same argument as above proves that $\tilde{Z}_\delta^{(L)}(w)\sqrt{\delta \kappa^L / r_L}$ converges in distribution to a standard Gaussian. Hence, since the factor multiplying $\tilde{Z}_\delta^{(L)}(w)$ is uniformly bounded below by 1 on A_δ^c,

$$\rho \left(\mathcal{L} \left(\tilde{Z}_\delta^{(L)}(w) \right), \sqrt{\delta^{-1} \kappa^{-L} r_L}\, \mathcal{N}(0,1) \right) \underset{k \to \infty}{\longrightarrow} 0 \,,$$

which again contradicts (3.6).

Finally, the proof of case c) is a variance computation: Indeed, note that in that case,

$$\text{Var } (\tilde{Z}_\delta^{(L)}(w)) = \frac{r_L}{\delta \kappa^L} \leq \frac{\delta_1}{\delta} \,.$$

In particular, with δ_0 denoting the atom at 0, since

$$\sup \left\{ \rho(\mu, \delta_0); \mu \text{ probabilitymeasure on } \mathbb{R}, \int x^2 d\mu < \frac{\delta_1}{\delta} \right\} \underset{\delta_1 \to 0}{\longrightarrow} 0 \,,$$

and

$$\sup\left\{\rho(\mathcal{N}(0,\sigma^2),\delta_0);\sigma^2<\frac{\delta_1}{\delta}\right\}\underset{\delta_1\to 0}{\longrightarrow}0\,,$$

we can choose δ_1 small enough (as function of ε_1 and nothing else) such that the triangle inequality yields a contradiction to (3.6). □

The next step involves transfering results from $\tilde{S}^{(L)}_{k_n}$ to $X_{\tau^{(L)}_{k_n}}$. Toward this end, define the random variable

$$W_L=W_L(w,\delta):=\frac{w\cdot\left(X_{\tau^{(L)}_{k_n}}-\tau^{(L)}_{k_n}v_L\right)}{\kappa^{-L}\sqrt{k_n}\sqrt{r_L\vee\delta\kappa^L}}\,.$$

Lemma 3.8 *Assume k_n such that (1.23) and (1.24) hold. Then,*

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\rho\left(\mathcal{L}\left(W_L\right),\mathcal{N}\left(0,\left(\mathbf{1}_{A_\delta(w)}+\mathbf{1}_{A^c_\delta(w)}\frac{r_L}{\delta\kappa^L}\right)\right)\right)=0\,. \tag{3.9}$$

Proof: Recall that

$$\tau^{(L)}_{k_n}=\kappa^{-L}\tilde{T}^{(L)}_{k_n}+\kappa^{-L}\sum_{i=1}^{k_n}\Delta^{(L)}_i\left(-\tilde{\tau}^{(L)}_i+Z^{(L)}_i\right), \tag{3.10}$$

and

$$X_{\tau^{(L)}_{k_n}}=\kappa^{-L}\tilde{S}^{(L)}_{k_n}+\sum_{i=L}^{k_n}\kappa^{-L}\Delta^{(L)}_i\left[-\tilde{X}^{(L)}_i+Y^{(L)}_i\right].$$

Since $|\tilde{X}^{(L)}_i|\le\tilde{\tau}^{(L)}_i$ and $|Y^{(L)}_i|\le Z^{(L)}_i$, Lemma 3.8 follows from Lemma 3.4 and [2] as soon as one shows that

$$\frac{\sum_{i=1}^{k_n}\Delta^{(L)}_i\tilde{\tau}^{(L)}_i}{\sqrt{k_n\kappa^L}}\overset{\text{Prob}}{\longrightarrow}0 \tag{3.11}$$

and

$$\sum_{i=1}^{k_n}\frac{\Delta^{(L)}_iZ^{(L)}_i}{\sqrt{k_n\kappa^L}}\overset{\text{Prob}}{\longrightarrow}0 \tag{3.12}$$

Let us prove (3.12), the proof of (3.11) being similar. We have

$$\begin{aligned}\overline{\mathbb{E}}^o\sum_{i=1}^{k_n}\Delta^{(L)}_iZ^{(L)}_i\sqrt{\frac{\kappa^{-L}}{k_n}}&\le\sqrt{\kappa^{-L}k_n}\sup_i\overline{\mathbb{E}}^o(\Delta^{(L)}_iZ^{(L)}_i)\le\sqrt{k_n\kappa^{-L}}\;\overline{\mathbb{E}}^o(\Delta^{(L)}_1)^{\frac{1}{\alpha'}}\sup_i\overline{\mathbb{E}}^o\left((\Delta^{(L)}_iZ^{(L)}_i)^\alpha\right)^{\frac{1}{\alpha}}\\&\le\sqrt{k_n\kappa^{-L}}\;M(L)^{\frac{1}{\alpha}}\phi'(L)^{\frac{1}{\alpha'}}\underset{L\to\infty}{\longrightarrow}0,\end{aligned}$$

by (1.24). □

We have completed the preliminaries to

Proof of Theorem 3 The main issue is to control the error between X_n and $X_{\tau^{(L)}_{k_n}}$. In view of (3.10), note that $\overline{\mathbb{E}}^o\tilde{T}^{(L)}_{k_n} = k_n\overline{\mathbb{E}}^o\tilde{\tau}^{(L)}_1 \geq k_n\frac{t_{av}}{2}$ and the argument in [2] (equation below (3.16)) shows that

$$\overline{\mathbb{E}}^o\frac{|\tau^{(L)}_{k_n} - \kappa^{-L}\tilde{T}^{(L)}_{k_n}|}{\kappa^{-L}\overline{\mathbb{E}}^o\tilde{T}^{(L)}_{k_n}} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{3.13}$$

Now, take $c_n = \kappa^{-2L}k_L[r_L \vee \delta\kappa^L]$. We start by showing that for all x,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\left|\overline{\mathbb{P}}^o\left(\frac{X_n\cdot w - nv_L\cdot w}{\sqrt{c_n}} \leq x\right) - \overline{\mathbb{P}}^o\left(\mathcal{N}\left(0, \left(\mathbf{1}_{A_\delta(w)}(n) + \mathbf{1}_{A^c_\delta(w)}(n)\frac{r_L}{\delta\kappa^L}\right)\right) \leq x\right)\right| = 0 \tag{3.14}$$

We have $\forall x, \delta_1, \delta_2$,

$$\begin{aligned}\overline{\mathbb{P}}^o\left(\frac{X_n\cdot w - nv_L\cdot w}{\sqrt{c_n}} \leq x\right) &\leq \overline{\mathbb{P}}^o\left(\left|\frac{\tau^{(L)}_{k_n}}{n} - 1\right| > \delta_2\right)\\ &+ \overline{\mathbb{P}}^o\left(\left|\frac{\tau^{(L)}_{k_n}}{n} - 1\right| < \delta_2, \frac{|X_n\cdot w - X_{\tau^{(L)}_{k_n}}\cdot w - nv_L\cdot w + \tau^{(L)}_{k_n}v_L\cdot w|}{\sqrt{c_n}} > \delta_1,\right)\\ &+ \overline{\mathbb{P}}^o\left(\frac{X_{\tau^{(L)}_{k_n}}\cdot w - \tau^{(L)}_{k_n}v_L\cdot w}{\sqrt{c_n}} \leq x + \delta_1\right) := \text{ I+II+III.}\end{aligned} \tag{3.15}$$

Using (3.13), (1.25), Lemma 3.1 and the estimates in Lemma 3.8, $I \to 0$ as $n \to \infty$. By Lemma 3.8, III $\to \Phi(\delta_1 + x)$. So, using a similar lower bound on the left most probability in (3.15), using the continuity of $\Phi(\cdot)$ and boundedness of the variance in (3.9), the claim (3.14) follows as soon as we prove that

$$\lim_{\delta_1\to 0}\ \limsup_{\delta_2\to 0}\ \limsup_{n\to\infty}\ \text{II} = 0\,.$$

Recall that $\overline{\mathbb{E}}^o\tilde{\tau}^{(L)}_1 \geq \frac{t_{av}}{2}$ for L large. Let $J = \left\{j : \left|\frac{j\kappa^{-L}\overline{\mathbb{E}}^o\tilde{\tau}^{(L)}_1}{n} - 1\right| < 2\delta_2\right\}$. (We have $k_n \in J$ for large n.)

Exactly as in the proof of (1.25),

$$p_0 \triangleq \overline{\mathbb{P}}^o\left(\exists j \notin J : \left|\frac{\tau^{(L)}_j}{n} - 1\right| < \delta_2\right) \underset{n\to\infty}{\longrightarrow} 0$$

$$\mathrm{II} \le p_0 + \overline{\mathbb{P}}^o \left(\max_{j \in J} \frac{1}{\sqrt{c_n}} \left| X_{\tau_j^{(L)}} \cdot w - X_{\tau_{k_n}^{(L)}} \cdot w - \tau_j^{(L)} v_L \cdot w + \tau_{k_n}^{(L)} v_L \cdot w \right| > \frac{\delta_1}{2} \right)$$
$$+ \overline{\mathbb{P}}^o \left(\max_{j \in J} \frac{1}{\sqrt{c_n}} \left| \tau_{j+1}^{(L)} - \tau_j^{(L)} \right| > \frac{\delta_1}{2} \right) := p_0 + p_1 + p_2 \,. \tag{3.16}$$

Concerning p_2, note that for n large, using (1.25)

$$|J| \le \frac{4\delta_2 n}{\kappa^{-L} \overline{\mathbb{E}}^o \tilde{\tau}_{k_n}^{(L)}} + 1 \le c_1 \delta_2 k_n$$

for some constant c_1. Hence,

$$\begin{aligned} p_2 \quad &\le \quad c_1 \delta_2 k_n \overline{\mathbb{P}}^o \left(\kappa^{-L} \tilde{\tau}_1^{(L)} \ge \frac{\delta_1 \sqrt{c_n}}{4} \right) + c_1 \delta_2 k_n \sup \overline{\mathbb{P}}^o \left(\Delta_1 Z_1^{(L)} \kappa^{-L} \ge \frac{\delta_1 \sqrt{c_n}}{4} \middle| \mathcal{F}_0^L \right) \\ &:= \quad p_{2,1} + p_{2,2} \end{aligned}$$

But

$$\begin{aligned} p_{2,1} &\le \frac{4^\alpha c_1 \delta_2 k_n}{\delta_1^\alpha c_n^{\frac{\alpha}{2}}} \kappa^{-\alpha L} \, \overline{\mathbb{E}}^o (\tilde{\tau}_1^{(L)})^\alpha \\ &\le \frac{c_2 \delta_2 M(L)}{\delta_1^\alpha [r_L \vee \delta \kappa^L]^{\alpha/2}} \frac{k_n \kappa^{-\alpha L}}{k_n^{\frac{\alpha}{2}} \kappa^{-\alpha L}} \le \frac{c_2 \delta_2}{\delta_1^\alpha \delta^{\alpha/2}} \frac{M(L)}{k_n^{\frac{\alpha}{2}-1} \kappa^{\alpha L/2}} \underset{L \to \infty}{\longrightarrow} 0 \end{aligned}$$

due to (1.23). Similarly, using [2, (3.12)]

$$p_{2,2} \le \frac{4^\alpha c_1 \delta_2 k_n}{\delta_1^\alpha c_n^{\frac{\alpha}{2}}} \kappa^{-\alpha L} M(L) \le c_3 \frac{\delta_2}{\delta_1^\alpha \kappa^{\alpha L/2}} k_n^{1-\frac{\alpha}{2}} M(L) \underset{L \to \infty}{\longrightarrow} 0$$

by (1.23)

It thus only remains to treat p_1. As above, we can replace $X_{\tau_j^{(L)}}$ and $\tau_j^{(L)}$ by $\kappa^{-L} \tilde{S}_j^{(L)}$ and $\kappa^{-L} \tilde{T}_j^{(L)}$, on a set which complement has probability smaller than

$$\overline{\mathbb{P}}^o (\max_{j \in J} \frac{\kappa^{-L}}{\sqrt{c_n}} \Delta_j^{(L)} |\tilde{X}_j^{(L)}| > \frac{\delta_1}{8}) + \overline{\mathbb{P}}^o (\max_{j \in J} \frac{\kappa^{-L}}{\sqrt{c_n}} \Delta_j^{(L)} |Y_j^{(L)}| > \frac{\delta_1}{8}) \,,$$

which tends to zero. Now, since $\tilde{S}_j^{(L)} - \tilde{T}_j^{(L)} v_L = \sum_{i=1}^j (\tilde{X}_j^{(L)} - \tilde{\tau}_j^{(L)} v_L)$ is a series of i.i.d. random variables, one has using Kolmogorov's inequality [4, Pg. 62] that

$$\overline{\mathbb{P}}^o \left(\sup_{j \in J} \frac{\kappa^{-L}}{\sqrt{c_n}} \left| \tilde{S}_j^{(L)} \cdot w - \tilde{S}_{k_n}^{(L)} \cdot w - \tilde{T}_j^{(L)} v_L \cdot w + \tilde{T}_{k_n}^{(L)} v_L \cdot w) \right| > \frac{\delta_1}{4} \right)$$
$$\le \frac{32 |J| \kappa^{-2L}}{\delta_1^2 c_n} r_L \le c_4 \frac{\delta_2}{\delta_1^2}$$

since $r_L \le 2\overline{\mathbb{E}}^o(\tilde{\tau}_1^{(L)})^2$ is bounded independently of L by A5. This completes the proof of (3.14).

We end by proving that (3.14) implies Theorem 3 with

$$v(n) = v_L \ , \quad R_n(w) = \frac{\kappa^{-L}}{\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)}} \operatorname{Var}\left(w \cdot [\tilde{X}_1^{(L)} - \tilde{\tau}_1^{(L)} v_L]\right) \quad \text{where } L = L(n) \ . \tag{3.17}$$

We argue by contradiction. If (1.26) does not hold for some w, take a subsequence n_k such that the left hand side is at least $\varepsilon > 0$. Moreover, going to a further subsequence if necessary, we can assume that $R_{n_k}(w)$ converges to a limit $R(w) \in [0, \infty]$. If $R(w)$ is positive and finite, this would contradict (3.14). If $R(w) = 0$, then $\kappa^{-L}\frac{\tilde{S}_{k_n} \cdot w - \overline{\mathbb{E}}^o \tilde{T}_{k_n}^{(L)} v_L \cdot w}{\sqrt{n}} \to 0$ in L^2, then $\frac{X_n \cdot w - n v_L \cdot w}{\sqrt{n}}$ would tend to 0 in probability, yielding another contradiction. Now, if $R(w) = \infty$, the two terms in Theorem (1.26) tend to the same limit $0, 1/2, 1$ according to $x < 0, x = 0, x > 0$, yielding again a contradiction. This proves part a) of Theorem (1.26).

Part b) of the Theorem follows by setting

$$L = L(n) := \left[\frac{\log n}{c + \log(1/\kappa)}\right]_{|\ell|_1} \ , \ k_n := \left[\frac{n}{\kappa^{-L(n)} \overline{\mathbb{E}}^o \tilde{\tau}_1^{(L(n))}}\right] \ ,$$

where $[z]_{|\ell|_1}$ denotes the largest element of $|\ell|_1 \mathbb{N}^*$ not larger than z. Then, k_n satisfy trivially (1.25), but also (1.23), (1.24) and both sequences tend to ∞. Indeed, $\overline{\mathbb{E}}^o \tilde{\tau}_1^{(L(n))} \ge t_{av}/2$, and

$$\limsup_{n \to \infty} \overline{\mathbb{E}}^o \tilde{\tau}_1^{(L)} = \limsup_{n \to \infty} \frac{\overline{\mathbb{E}}^o \tilde{X}_1^{(L)} \cdot \ell}{v \cdot \ell}$$

(since the ratio of the two expectations tends to $v \cdot \ell$, see sentence before Theorem 3.17 in [2]), where $\tilde{X}_1^{(L)} \cdot \ell$ has exponential tails by Lemma (5.3) in [2].

□

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