LARGE DEVIATIONS OF EMPIRICAL MEASURES OF ZEROS OF RANDOM POLYNOMIALS

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Abstract. We prove a large deviation principle for empirical measures

$$Z_s := \frac{1}{N} \sum_{\zeta : s(\zeta)=0} \delta_\zeta, \quad (N := \#\{\zeta : s(\zeta) = 0)\}$$

of zeros of random polynomials in one variable. By random polynomial, we mean a Gaussian measure on the space $\mathcal{P}_N = H^0(\mathbb{CP}^1, \mathcal{O}(N))$ determined by inner products $G_N(h, \nu)$ induced by any smooth Hermitian metric h on $\mathcal{O}(1) \to \mathbb{CP}^1$ and any probability measure $d\nu$ on $\mathbb{CP}^1$ satisfying the weighted Bernstein-Markov inequality. The speed of the LDP is N^2 and the rate function is closely related to the weighted energy of probability measures on $\mathbb{CP}^1$, and in particular its unique minimizer is the weighted equilibrium measure.

1. Introduction and statement of results

The purpose of this article is to establish a large deviations principle for the empirical measure

$$Z_s := d\mu_\zeta := \frac{1}{N} \sum_{\zeta : s(\zeta)=0} \delta_\zeta, \quad N := \#\{\zeta : s(\zeta) = 0\} \tag{1}$$

of zeros of a random polynomial s of degree N. Here, δ_ζ is the Dirac point measure at $\zeta \in \mathbb{C}$. We define random polynomials of degree N by putting geometrically defined Gaussian probability measures $d\gamma_N$ on the space $\mathcal{P}_N$ of holomorphic polynomials of degree N, or equivalently, Fubini-Study measures dV_N^{FS} on the projective space $\mathbb{P}\mathcal{P}_N$ of polynomials (see §2 for background). The measures $d\gamma_N, dV_N^{FS}$ are determined by a pair $(h = e^{-\varphi}, \nu)$ consisting of a 'weight' φ or (globally) a Hermitian metric h on the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{CP}^1$, and a probability measure ν on $\mathbb{CP}^1$ satisfying the Bernstein-Markov condition (11). The Gaussian measure on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ and the Fubini-Study measure on $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ are induced from the Hermitian inner products

$$||s||^2_{G_N(h,\nu)} := \int_{\mathbb{C}} \|s(z)\|^2_{h^N} d\nu(z), \quad (s \in \mathcal{P}_N). \tag{2}$$

The zeros then become equidistributed with high probability in the large N limit according to an equilibrium measure $d\nu_{h,K}$ depending on h and the support K of ν, which reflects the competition between the repulsion of nearby zeros and the force of the external electric field (curvature form) ω_h of h (see [SZ, Ber1, Ber2, BB]). The large deviations results show that the empirical measures (1) are concentrated exponentially closely (with speed N^2) to $\nu_{h,K}$ as $N \to \infty$, with rate given by a rate function $\tilde{I}^{h,K}$ that is minimized by $\nu_{h,K}$.

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The large deviations rate function is determined from the joint probability density $D_N(\zeta_1, \ldots, \zeta_N)$ of zeros, which measures the likelihood of a given configuration of N points arising as zeros of $s \in \mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$. The joint probability density is the density of a joint probability current on the configuration space

$$(\mathbb{CP}^1)^{(N)} = Sym^N \mathbb{CP}^1 := \underbrace{\mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1}_{N} / S_N$$

of N points of $\mathbb{CP}^1$. Here, S_N is the symmetric group on N letters. The joint probability current is by definition the pushforward

$$\vec{K}_n^N(\zeta_1, \ldots, \zeta_N) := \mathcal{D}_* dV_N^{FS} \tag{3}$$

of the Fubini-Study measure on $\mathbb{P}\mathcal{P}_N$ under the 'zero set' or divisor map

$$\mathcal{D} : \mathcal{P}_N \to (\mathbb{CP}^1)^{(N)}, \quad \mathcal{D}(s) = \zeta_1 + \cdots + \zeta_N,$$

where $\{\zeta_1, \ldots, \zeta_N\}$ is the zero set of s. Following a standard notation in algebraic geometry, we are writing an unordered set of points $\{\zeta_1, \ldots, \zeta_N\}$ as a formal sum (i.e. a divisor) $\zeta_1 + \cdots + \zeta_N \in (\mathbb{CP}^1)^{(N)}$. In the case of polynomials, $\mathcal{D}$ is obviously surjective (any N-tuple of points is the zero set of some polynomial of degree N); one may identify $\mathbb{P}\mathcal{P}_N \simeq (\mathbb{CP}^1)^{(N)}$.

The zero set can also be encoded by the probability measure (1) on $\mathbb{CP}^1$. This identification defines a map

$$\mu : (\mathbb{CP}^1)^{(N)} \to \mathcal{M}(\mathbb{CP}^1), \quad d\mu_{\zeta_1 + \cdots + \zeta_N} = d\mu_\zeta := \frac{1}{N} \sum_{j=1}^N \delta_{\zeta_j},$$

where $\mathcal{M}(\mathbb{CP}^1)$ is the (Polish) space of probability measures on $\mathbb{CP}^1$, equipped with the weak-* topology (i.e. the topology induced by weak, or equivalently vague, convergence of measures). In general, for any closed subset $F \subset \mathbb{CP}^1$ we denote by $\mathcal{M}(F)$ the probability measures supported on F. Thus, the zero sets can all be embedded as elements of the space $\mathcal{M}(\mathbb{CP}^1)$ of probability measures on $\mathbb{CP}^1$. This point of view is ideal for taking large N limits, and has been previously used in many similar situations, for instance in analyzing the eigenvalues of random matrices [BG, BZ, HP].

Under the map $s \to \mu \circ \mathcal{D}(s)$ we further push forward the joint probability current to obtain a probability measure

$$\mathbf{Prob}_N = \mu_* \mathcal{D}_* dV_N^{FS} \tag{4}$$

on $\mathcal{M}(\mathbb{CP}^1)$. Our main results show that this sequence of measures $\mathbf{Prob}_N$ satisfies a large deviations principle with speed N^2 and with a rate function $\tilde{I}^{h,K}$ reflecting the choice of (h, ν). Roughly speaking this means that for any Borel subset $E \subset \mathcal{M}(\mathbb{CP}^1)$,

$$\frac{1}{N^2} \log \mathbf{Prob}_N\{\sigma \in \mathcal{M} : \sigma \in E\} \to -\inf_{\sigma \in E} \tilde{I}^{h,K}(\sigma).$$

Before stating our results, we recall some notation and background. Throughout this article we use the language of complex geometry, and in particular we identity $\mathcal{P}_N = H^0(\mathbb{CP}^1, \mathcal{O}(N))$, i.e. we identify polynomials of degree N with holomorphic sections of the Nth power of the hyperplane line bundle [GH] (Chapter I.3). In the affine chart $U = \mathbb{CP}^1 \backslash \{\infty\}$, and in the standard holomorphic frame $e : U \to \mathbb{CP}^1$, the Hermitian metric h is represented by the function $||e||_h^2 = e^{-\varphi}$.

In weighted potential theory, the function φ is referred to as a weight [ST, B, B2]. Several authors have generalized weighted potential theory to Kähler manifolds, and we use their geometric language [GZ, Ber1, Ber2, B, B2, BS]. The key point is that our weights $e^{-\varphi}$ are local expressions for global smooth Hermitian metrics h on $\mathcal{O}(1) \to \mathbb{CP}^1$ and therefore have special behavior at ∞. We use their associated Green's functions $G_h(z, w)$ to define the basic objects of potential theory: potentials, energies and capacities. The resulting Green's function has the standard logarithmic $-\infty$ singularity on the diagonal, but is bounded above (unlike the logarithmic kernel $\log|z - w|$ on $\mathbb{C}$). To be precise, let ω_h be the curvature $(1, 1)$ form of a smooth Hermitian metric h on $\mathbb{CP}^1$. The Green's function G_h relative to ω_h is defined to be the unique solution $G_h(z, \cdot) \in \mathcal{D}'(\mathbb{CP}^1)$ of

$$\begin{cases} (i) \quad dd^c_w G_h(z,w) = \delta_z(w) - \omega_h(w), \\ (ii) \quad G_h(z,w) = G_h(w,z), \\ (iii) \quad \int_{\mathbb{CP}^1} G_h(z,w)\omega_h(w) = 0, \end{cases} \tag{5}$$

where the equality in the top line is in the sense of $(1, 1)$ forms. Existence of G_h is guaranteed by the $\partial\bar{\partial}$ Lemma even when ω_h is non-positive, i.e. is not a Kähler form; uniqueness follows from condition (iii). As shown in Lemma 8 of §4, in the frame $e(z)$ over the affine chart $\mathbb{C}$ in which $h = e^{-\varphi}$ and $\omega_h = dd^c\varphi$, the Green's function has the local expression,

$$G_h(z, w) = 2\log|z - w| - \varphi(z) - \varphi(w) + E(h), \tag{6}$$

where

$$E(h) := \left(\int_{\mathbb{CP}^1} \varphi(z)\omega_h + 4\pi\rho_\varphi(\infty) \right), \tag{7}$$

ρ_φ being a certain Robin constant (see (63) of §8.2, and also (38)). The constant $E(h)$ plays a role in our large deviation rate functional. The Green's potential of a measure μ (with respect to ω_h) is defined by

$$U^\mu_h(z) = \int_{\mathbb{CP}^1} G_h(z, w)d\mu(w), \tag{8}$$

and the Green's energy by

$$\mathcal{E}_h(\mu) = \int_{\mathbb{CP}^1\times\mathbb{CP}^1} G_h(z, w)d\mu(z)d\mu(w)\,. \tag{9}$$

We now introduce our Gaussian random ensembles and the main assumption on the measures ν underlying our inner products. In the local frame any holomorphic section may be written $s = fe$ where $f \in \mathcal{O}(U)$ is a local holomorphic function. The inner product (2) then takes the form,

$$||s||_{G_N(h,\nu)} = \int_{\mathbb{C}} |f(z)|^2 e^{-N\varphi} d\nu(z). \tag{10}$$

The measure ν is assumed to satisfy two conditions. The first is the weighted Bernstein-Markov condition (see [B2] (3.2) or [BB], Definition 4.3 and references):

For all $\epsilon > 0$ there exists $C_\epsilon > 0$ so that

$$\sup_K \|s(z)\|_{h^N} \le C_\epsilon e^{\epsilon N} ||s||_{G_N(h,\nu)}\,, \quad s \in H_0(\mathbb{CP}^1, \mathcal{O}(N)). \tag{11}$$

Here, and throughout this article, we write

$$K = \quad \text{supp}\ \nu. \tag{12}$$

We further assume that

$$K \ is \ non-h-thin \ at \ all \ of \ its \ points \tag{13}$$

in the sense that K is non-thin at x (with respect to G_h-potentials U_h^μ) for all $x \in K$ (see §5.3 for the definition). For the purposes of this paper, the important property of K is that, for any $z^* \in \partial K$, the capacity $\text{Cap}_h(D(z^*, \epsilon) \cap K) > 0$ for every $\epsilon > 0$, where Cap_h is the Green's capacity (see (51)). Here, $D(z^*, \epsilon)$ is a metric disc of radius ϵ; the condition is independent of the choice of metric. For example, any connected set with more than one point is non-thin at any point of its closure (see [Ran], Theorem 3.8.3). We refer to §5.3 for the relevant results on thinness and capacity.

We then define the Gaussian probability measures $\gamma_{h^N,\nu}$ on $\mathcal{P}_N = H^0(\mathbb{CP}^1, \mathcal{O}(N))$ as the Gaussian measure determined by the inner product (2) (the definition is reviewed in §2.3). The associated Fubini-Study measures are denoted by $dV^{FS}_{h^N,\nu}$ on $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ (see §3 for the definition).

We will see, c.f. Lemma 25 and Proposition 26, that under Assumptions (11) and (13), for any $\mu \in \mathcal{M}(\mathbb{CP}^1)$, $\mathcal{E}_h(\mu) < \infty$ and $|\sup_K U_h^\mu| < \infty$. In particular, the function

$$I^{h,K}(\mu) = -\frac{1}{2}\mathcal{E}_h(\mu) + \sup_K U_h^\mu, \quad \mu \in \mathcal{M}(\mathbb{CP}^1) \tag{14}$$

is well-defined (with $+\infty$ as possible value). Set

$$E_0(h) = \inf_{\mu \in \mathcal{M}(\mathbb{CP}^1)} I^{h,K}(\mu), \quad \tilde{I}^{h,K} = I^{h,K} - E_0(h)\,. \tag{15}$$

The infimum $\inf_{\mu \in \mathcal{M}(\mathbb{CP}^1)} I^{h,K}(\mu)$ is achieved at the Green's equilibrium measure $\nu_{h,K}$ with respect to (h, K), and $E_0(h) = \frac{1}{2}\log \text{Cap}_h(K)$, where (as above) $\text{Cap}_h(K)$ is the Green's capacity with respect to h. See Lemma 4 (the lemma is proved in §7.4). By the Green's equilibrium measure we mean the minimizer of $-\mathcal{E}_h$ on $\mathcal{M}(K)$. We refer to §5 for definitions and discussion of $\nu_{h,K}$ and of $\text{Cap}_h(K)$ (see (51)).

1.1. **Statement of results.** Our main result is the following:

THEOREM **1.** *Let h be a smooth Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^1$ and let $d\nu \in \mathcal{M}(\mathbb{CP}^1)$ satisfy the Bernstein-Markov property (11) and the nowhere thinness assumption (13). Then $\tilde{I}^{h,K}$ of* (15) *is a strictly convex rate function and the sequence of probability measures $\{\mathbf{Prob}_N\}$ on $\mathcal{M}(\mathbb{CP}^1)$ defined by (4) satisfies a large deviations principle with speed N^2 and rate function $\tilde{I}^{h,K}$ (see* (16) *below). Further, there exists a unique measure $\nu_{h,K} \in \mathcal{M}(\mathbb{CP}^1)$ minimizing $\tilde{I}^{h,K}$, namely the Green's equilibrium measure of K with respect to h.*

(Recall, see [DZ, Pg. 4], that a function $I : \mathcal{M}(\mathbb{CP}^1) \to \mathbb{R}$ is a rate function if it is lower semicontinuous and non-negative.)

Theorem 1 shows that the empirical measures $d\mu_\zeta$, see (1), concentrate near $\nu_{h,K}$ at an exponential rate. More precisely, if $B(\sigma, \delta)$ denotes the ball of radius δ around $\sigma \in \mathcal{M}(\mathbb{CP}^1)$ in the Wasserstein metric, and $B^o(\sigma,\delta)$ (respectively, $\overline{B(\sigma,\delta)}$) denote its interior (respectively, its closure), then

$$\begin{aligned} -\inf_{\mu\in B^o(\sigma,\delta)} \tilde{I}^{h,K}(\mu) &\leq \liminf_{N\to\infty} \tfrac{1}{N^2}\log \mathbf{Prob}_N(B(\sigma,\delta)) \\ &\leq \limsup_{N\to\infty} \tfrac{1}{N^2}\log \mathbf{Prob}_N(B(\sigma,\delta)) \leq -\inf_{\mu\in\overline{B(\sigma,\delta)}} \tilde{I}^{h,K}(\mu)\,. \end{aligned} \tag{16}$$

(With $\mathrm{Lip}(\mathbb{CP}^1)$ denoting the space of Lipschitz functions on $\mathbb{CP}^1$ with Lipschitz constant 1, the Wasserstein metric on $\mathcal{M}(\mathbb{CP}^1)$ is defined as

$$d_W(\mu,\mu') = \sup_{f\in \mathrm{Lip}(\mathbb{CP}^1)} \int_{\mathbb{CP}^1} f d(\mu-\mu')\,.$$

It is compatible with the topology of weak convergence in $\mathcal{M}(\mathbb{CP}^1)$.)

Theorem 1 implies afortiori that the expected value of $d\mu_\zeta$ tends to $\nu_{h,K}$, refining the result of [SZ] on the equilibrium distribution of zeros in the unweighted case and the more general results in the subsequent articles [B, BS, Ber1, Ber2], when restricted to the univariate setup under discussion in this paper. Intuitively, in the unweighted case, zeros repel each other like electrons to the outer boundary of K. A Hermitian metric or weight $h = e^{-\varphi}$ with $\omega_\varphi > 0$ behaves like an uphill potential which pushes electrons back into the interior of K and gives rise to an equilibrium potential which charges the interior of K, with extra accumulation along ∂K.

The inner product (2) depends only on the restriction of the metric h to K, see (12), and consequently the rate function should only depend on this restriction. To see this, we rewrite it in the standard affine chart $\mathbb{C}$ and frame for $\mathcal{O}(1)$ in the form

$$-\frac{1}{2}\mathcal{E}_h(\mu) + \sup_K U_h^\mu = -\Sigma(\mu) + \sup_{z\in K}\{2\int_{\mathbb{C}} \log|z-w| d\mu(w) - \varphi(z)\}, \tag{17}$$

where

$$\Sigma(\mu) = \int_{\mathbb{C}\times\mathbb{C}} \log|z-w| d\mu(z) d\mu(w) \tag{18}$$

is the logarithmic energy or entropy function. In the large deviations analysis, it is more convenient to use the formulation in Theorem 1 which uses the 'compactification' of the metric to $\mathbb{CP}^1$.

1.2. **Examples.** As an illustration of the methods and results, we observe that Theorem 1 applies to the Kac-Hammersley ensemble as in [SZ], where $d\nu = \delta_{S^1}$ (the invariant probability measure on the unit circle), and where the weight $e^{-\varphi} = 1$. Hence, the inner product is simply $\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$. It is simple to verify that $d\nu = \delta_{S^1}$ satisfies the Bernstein-Markov property, i.e. that for holomorphic polynomials of degree N, $||p_N||_{S^1} \leq C_\epsilon e^{\epsilon N}\frac{1}{2\pi}\left(\int |p_N(e^{i\theta})|^2 d\theta\right)^{1/2}$. Indeed, we let $\Pi_N(z,w) = \sum_0^N z^n \bar{w}^n$ denote the Szegö reproducing kernel for $\mathcal{P}_N$ with this measure. Then by the Schwarz inequality,

$$\sup_{z\in S^1} |p_N(z)| \leq \sup_{z\in S^1} \sqrt{\Pi_N(z,z)}||p_N||_{L^2(\nu)} \leq \sqrt{N}||p_N||_{L^2(\nu)}.$$

On S^1 we are taking the weight to be 'flat', i.e. the Hermitian metric to be $\equiv 1$. We are free to choose a smooth extension of this Hermitian metric to $\mathcal{O}(1)\to\mathbb{CP}^1$. For instance, we may take $h = e^{-\varphi}$ to be S^1 invariant, equal 1 in a neighborhood of $\mathbb{CP}^1$ and to equal the Fubini-Study metric in a neighborhood of ∞. There is of course no unique choice of the smooth extension. With any of these extensions, δ_{S^1} is easily seen to satisfy the condition (13).

At the opposite extreme, the methods and results apply to the case where $d\nu = \omega_{FS}$, the Fubini-Study Kähler form, and where $h = h_{FS} = e^{-\log(1+|z|^2)}$. The Bernstein-Markov property follows from the same calculation as in the Kac-Hammersley example, except that the Szegö reproducing kernel is different (but still equals $N+1$ on the diagonal; see [SZ, SZ2, SZ3] for further background). The regularity condition is obviously satisfied.

1.3. **An application - hole probabilities.** The large deviation results give an accurate upper bound for 'hole probabilities' for our ensembles of Gaussian random polynomials of one complex variable. A hole probability for an open set U is the probability that the random polynomial has no zeros in U. Large deviations estimates for hole probabilities for balls $U = B_R$ of increasing radius were proved in [SoTs] for certain random analytic functions. More in line with the present paper are asymptotic hole probabilities as the degree $N \to \infty$ of random holomorphic sections of powers $L^N \to M$ of positive line bundles in [SZZr]. The results there hold in all dimensions, but the stronger assumption is made that h is a Hermitian metric with positive curvature $(1,1)$ form.

We now state a hole probability for our general Gaussian ensembles on $\mathbb{CP}^1$, where the hole is an open set $U \subset \mathbb{CP}^1$. We consider the

$$A_U = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(U) = 0\}\,.$$

The following hole probability has the same speed of exponential decay as in [SZZr].

Corollary **1.** *For any of the Gaussian ensembles $G_N(h,\nu)$ and for any open set U,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_U) \le -\inf_{\mu\in A_U} \tilde{I}^{h,K}(\mu)\,.$$

Proof. If $\mu_n \to \mu$ weakly in $\mathcal{M}(\mathbb{C})$ then $\liminf_{n\to\infty} \mu_n(U^c) \le \mu(U^c)$. Thus, A_U is a closed set, both in $\mathcal{M}(\mathbb{C})$ and (with a slight abuse of notation) in $\mathcal{M}(\mathbb{CP}^1)$. The upper bound is then immediate from Theorem 1.

□

Unfortunately, the large deviation principle is not quite strong enough to provide complementary lower bounds. Indeed, the set

$$A_U^o = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(U) = 1\}$$

has empty interior for any set $U \neq \mathbb{CP}^1$, and the large deviations lower bound is $-\infty$. The best one can obtain from the LDP is that, with U closed and the set

$$A_U^{p,o} = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(U) > p\}\,,$$

one has by a similar analysis

$$\lim_{p\nearrow 1} \liminf_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_U^{p,o}) \ge -\inf_{\mu\in A_U^o} \tilde{I}^{h,K}(\mu)\,.$$

The constrained infimum $\inf_{\mu\in A_U} \tilde{I}^{h,K}(\mu)$ is achieved by a measure $\nu_{U,h,K}$, which may be regarded as a relative weighted equilibrium measure with respect to the two independent sets

U, K. In general it is impossible to evaluate numerically. In a special case, we can however evaluate it. With $r < 1$, let $U^c = \bar{B}_r \subset \mathbb{C}$ be the closed ball of radius r centered at the origin. Set

$$A_r = \{\mu \in \mathcal{M}(\mathbb{C}) : \mu(\bar{B}_r) = 1\}\,.$$

Corollary **2.** *For the Kac-Hammersley ensemble, and for $r < 1$, we have*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_r) \le \log r\,.$$

Proof. By Corollary 1,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(A_r) \le -\inf_{\mu\in A_r} \tilde{I}^{h,K}(\mu)\,. \tag{19}$$

We specialize the last expression in the case of the Kac-Hammersley ensemble: written in the affine chart around 0, we have from (14), (15) and (17)

$$\tilde{I}^{h,K}(\mu) = -\Sigma(\mu) + 2\sup_{z\in S^1}\int_{\mathbb{C}} \log|z-w| d\mu(w)\,,$$

where we used that for any $R \ge 1$,

$$\inf_{\mu\in A_R} I^{h,K}(\mu) = -\Sigma(\nu) + 2\int_{\mathbb{C}} \log|1-w| d\nu(w) = 0\,,$$

with $\nu = \delta_{S^1}$ the uniform distribution on S^1.

Fix $r < 1$. For given $\mu \in A_r$, let $\tilde{\mu}$ denote the radial symmetrization of μ, that is, for any measurable $A \subset \mathbb{C}$,

$$\tilde{\mu}(A) = \frac{1}{2\pi}\int_0^{2\pi}\int \mathbf{1}_{ze^{i\theta}\in A} d\mu(z) d\theta\,.$$

Due to the convexity of A_r and of $I^{h,K}(\cdot)$, the minimizer μ^* in the right side of (19) is radially symmetric, i.e. $\tilde{\mu}^* = \mu^*$, and belongs to A_r. Using the identity, valid for any $s \le 1$,

$$\int_0^{2\pi} \log|1 - se^{i\theta}| d\theta = 0\,, \tag{20}$$

we thus obtain

$$\inf_{\mu\in A_r} \tilde{I}^{h,K}(\mu) = [\inf_{\mu\in A_r, \mu=\tilde{\mu}} -\Sigma(\mu)] + \Sigma(\delta_{S^1}) = \inf_{\mu\in A_r,\mu=\tilde{\mu}} -\Sigma(\mu)\,.$$

For $\mu \in A_r$ with $\mu = \tilde{\mu}$, write $\mu = \rho(dr) \times d\theta$, with $\rho \in \mathcal{M}([0, r])$. Then,

$$\begin{aligned}
\Sigma(\mu) &= \frac{1}{4\pi^2}\iint_0^{2\pi}\iint_0^{r} \log|se^{i\theta} - s'e^{i\theta'}|\rho(ds)\rho(ds') \\
&= \frac{1}{2\pi}\int_0^{2\pi}\iint_0^{r} \log|s - s'e^{i\theta'}|\rho(ds)\rho(ds') \\
&= -\iint_0^{r} (\log s')[2\mathbf{1}_{s'>s} + \mathbf{1}_{s=s'}]\rho(ds)\rho(ds')\,,
\end{aligned}$$

where we used (20) in the last equality. The last expression is maximized (over $\rho \in \mathcal{M}([0, r])$) at $\rho_r = \delta_r$.

□

1.4. **Discussion of the proof.** Functions somewhat similar to (14) or (17) arise as rate functions in large deviations problems for empirical measures of eigenvalues of random matrices (see e.g. [BG, BZ]). In particular, much of the analysis of the energy term $\mathcal{E}_h(\mu)$ can be carried over from the eigenvalue setting and from known results in classical weighted potential theory on $\mathbb{C}$ (see [ST]). We recall that the weighted equilibrium measure of K with respect to the weight e^{-Q} is the unique maximizer in $\mathcal{M}(K)$ of the weighted energy function,

$$\Sigma_{Q,K}(\mu) = \int_K \int_K \log \left(|z-w| e^{-Q(z)} e^{-Q(w)} \right) d\mu(w) d\mu(z). \tag{21}$$

We observe that $Q = \frac{\varphi}{2}$ in the global setting, i.e. the weight is essentially a Hermitian metric.

However, the (non-differentiable) sup function $J^{h,K}(\mu) := \sup_K U_h^\mu$ is quite different from, and somewhat more difficult than, the linear functions such as $\int x^2 d\mu$ which occur in the eigenvalue setting. Under Assumption (13), we show that it is convex and continuous on $\mathcal{M}(\mathbb{CP}^1)$ with respect to weak convergence (which, due to the compactness of $\mathbb{CP}^1$, is equivalent to vague convergence) of probability measures (Lemma 26). The continuity also uses the fact that the Green's function G_h is bounded above on $\mathbb{CP}^1$.

It is not obvious that the minimizer of $I^{h,K}$ should be the same as the maximizer of (21). This is proved in Proposition 29. The main differences are that (i) $\mathcal{E}_h$ is not constrained to measures supported on K; (ii) the second $\sup_K$ term is additional to the energy. In Proposition 29, we show that $\nu_{h,K}$ minimizes both $-\mathcal{E}_h$ and $J^{h,K}$.

Besides potential theory, an important ingredient in the proof of Theorem 1 is a formula for the joint probability current of zeros when $\mathcal{P}_N$ is endowed with the Gaussian measure derived from the inner product (2). A novelty of our presentation is that we derive the joint probability current in a natural way from the associated Fubini-Study probability measure on $\mathbb{P}\mathcal{P}_N$. In the following, we work in the standard chart, i.e. $(\mathbb{C})^{(N)}$. Let $\Delta(\zeta) = \prod_{i<j}(\zeta_i - \zeta_j)$ denote the Vandermonde determinant, $s_\zeta(\cdot) \in \mathcal{P}_N$ the polynomial with zero set $\{\zeta_1, \dots, \zeta_N\}$, and $d^2\zeta = d\zeta \wedge d\bar{\zeta}$ on $\mathbb{C}$.

In the following, and hereafter, we often use the following identity:

$$\int_{\mathbb{CP}^1} G_h(z,w) dd^c \log ||s_\zeta(w)||^2_{h^N} = N \int G_h(z,w) d\mu_\zeta(w) = N U_h^{\mu_\zeta}(z), \tag{22}$$

which follows from the definitions (1) (5), (8) and from the Poincaré-Lelong formula (2.1).

Proposition **3.** *The joint probability current (3) (see also (34)) is given in the affine chart* $(\mathbb{C})^N \subset (\mathbb{CP}^1)^N$ *by*

$$\vec{K}_n^N(\zeta_1, \dots, \zeta_N) = \frac{1}{Z_N(h)} \frac{|\Delta(\zeta_1, \dots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N}{\left(\int_{\mathbb{CP}^1} \prod_{j=1}^N |(z-\zeta_j)|^2 e^{-N\varphi(z)} d\nu(z) \right)^{N+1}} \tag{23}$$

$$= \frac{1}{\hat{Z}_N(h)} \frac{\exp\left(\sum_{i<j} G_h(\zeta_i, \zeta_j) \right) \prod_{j=1}^N e^{-2N\varphi(\zeta_j)} d^2\zeta_j}{\left(\int_{\mathbb{CP}^1} e^{N \int_{\mathbb{CP}^1} G_h(z,w) d\mu_\zeta} d\nu(z) \right)^{N+1}}. \tag{24}$$

where G_h is the Green's function (5). Also, $Z_N(h)$ and $\hat{Z}_N(h)$ are normalizing constants, defined so that the measure on the left side has mass one.

The second expression (24) is invariantly defined. We will see that

$$\begin{cases} Z_N(h) = |\det \mathcal{A}_N(h)|^{-2} = \mathrm{Vol}(\mathcal{B}^2(L^N, h^N)), & \text{see (32)} \\ \hat{Z}_N(h) = |\det \mathcal{A}_N(h)|^{-2} e^{-(-\frac{1}{2}N(N-1)+N(N+1))E(h)}, & \text{see (7)}. \end{cases} \tag{25}$$

Here, $\mathcal{A}_N(h)$ is the change of basis matrix from the monomials z^j to an orthonormal basis for the inner product $G_N(h,\nu)$ on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$, $\mathcal{B}^2(L^N, h^N) = \{s \in H^0(\mathbb{C}, L^N) : \|s\|^2_{h^N} \le 1\}$ is the unit ball in $H^0(\mathbb{C}, L^N)$ and Vol is with respect to the Lebesgue measure. In Lemma 18, we further rewrite the expression for $\vec{K}^N_n(\zeta_1, \ldots, \zeta_N)$ as a functional $I_N(\mu_\zeta) = I^{h,\nu}_N(\mu_\zeta)$ on the measures μ_ζ. The rate function I is then extracted from I_N as $N \to \infty$.

To complete the calculation, we determine the logarithmic asymptotics of $\hat{Z}_N(h)$.

Lemma **4.** *We have,*

$$\lim_{N\to\infty} \frac{1}{N^2} \log \hat{Z}_N(h) = \frac{1}{2} \log \mathrm{Cap}_h(K)\,.$$

This limit formula gives an alternative approach to the asymptotics of $|\det \mathcal{A}_N(h)|^2$ from that in [BB], in this one dimensional setting.

1.5. **Sketch of proof for the Kac-Hammersley ensemble.** We now sketch the proof of Theorem 1 in the case of the Kac-Hammersley ensemble. In this case, we do not need the geometric language used in the rest of the paper.

Consider the polynomial $P_N(z) = \sum_{i=0}^N a_i z^i = a_N[z^N + \sum_{i=0}^{N-1} b_i z^i]$, where the a_i are independent Gaussian circular (i.e., zero mean and i.i.d. real and imaginary part) standard complex random variables and $b_i = a_i/a_N$. We have $P_N(z) = a_N \prod_{i=1}^N (z - z_i)$. Further, conditioned on a_N, the variables $\{b_i\}_{i=0}^{N-1}$ are independent, Gaussian circular, of zero mean and variance $|a_N|^{-2}$.

Let $\Delta = \prod_{i<j} |z_i - z_j|$. Then, the Jacobian of the transformation $\{b_i\}_{i=0}^{N-1} \mapsto \{z_i\}_{i=1}^N$ is $|\Delta|^2$. On the other hand, with $d\mu_\xi \in \mathcal{M}(\mathbb{C})$ denoting the empirical measure of the zeros of P_N,

$$\begin{aligned} & |a_N|^2 + \sum_{i=0}^{N-1} |a_N b_i|^2 = \sum |a_i|^2 \\ = \;& (2\pi)^{-1} \int_{S^1} P_N(z) P_N^*(z) dz \\ = \;& \frac{|a_N|^2}{2\pi} \int_{S^1} \prod |z - z_i|^2 dz = \frac{|a_N|^2}{2\pi} \int_{S^1} e^{2N \int \log|z-x| d\mu_\xi(x)} dz := |a_N|^2 e^{N \mathcal{J}_N(\mu_\xi)}\,, \end{aligned}$$

where the integrals are path integral along the unit circle, and we used the fact that the integrand is real to express it as an exponential of a real function. We then have, for any measurable set $A \subset \mathcal{M}(\mathbb{C})$,

$$\begin{aligned} \mathbf{Prob}_N(d\mu_\xi \in A) &= \frac{1}{Z_N} \int_{z_i:\Delta\neq 0} dz_1 \ldots dz_N \mathbf{1}_{\{L_N \in A\}} \int_0^\infty y^N e^{N^2 \Sigma(L_N) - y e^{N \mathcal{J}_N(L_N)}} dy \\ &= \frac{1}{\widetilde{Z_N}} \int_{z_i:\Delta\neq 0} dz_1 \ldots dz_N \mathbf{1}_{\{L_N \in A\}} e^{N^2 \Sigma(L_N)} e^{-N^2 \mathcal{J}_N(L_N)}. \end{aligned}$$

Here, $L_N = N^{-1}\sum_{i=1}^N \delta_{z_i}$, Σ is as in (18) (with the convention $\log(0) = 0$), and Z_N, $\widetilde{Z_N}$ are normalization constants.

Now, for each fixed μ for which $\log|z - \cdot|$ is uniformly integrable for $z \in S^1$, we have that

$$\mathcal{J}_N(\mu) = N^{-1}\log\left(\frac{1}{2\pi}\int_{S_1}\exp(2N\langle\mu, \log|z-\cdot|\rangle)dz\right) \to_{N\to\infty} 2J(\mu)\,,$$

where

$$J(\mu) := \max_{z\in S^1}\int \log|z-x|d\mu(x)\,.$$

One thus expects, as in [BG, BZ], that for "nice" sets A,

$$N^{-2}[\log \mathbf{Prob}_N(d\mu_\xi \in A) + \log \widetilde{Z}_N] \to \inf_{\mu\in A}\left(2J(\mu) - \Sigma(\mu)\right)\,.$$

Thus, it is natural to expect to obtain the large deviation principle, with speed N^2 and rate function

$$2J(\mu) - \Sigma(\mu) - \inf_{\nu\in\mathcal{M}(\mathbb{C})}[2J(\nu) - \Sigma(\nu)]\,.$$

(Compare with (17), noting that $K = S^1$ and $\varphi = 1$ on S^1 for the Kac-Hammersley ensemble.) The technical details of the derivation, however, are best handled in a more general geometric framework, where relevant properties of the rate function are more transparent.

1.6. **Generalizations.** We close the introduction with some comments on the generalization of the results of this article to other Kähler manifolds. In the sequel [Z] we use the method of this article to give an explicit formula for the joint probability current in the more difficult higher genus cases[1]. In higher genus, the relation between configuration spaces and sections of line bundles of degree N is the subject of Abel-Jacobi theory, and the formula for the joint probability current involves such objects as the prime form. Some of the geometric discussion of this paper is intended to set the stage for the higher genus sequel. The results can also be generalized from Gaussian ensembles to non-linear ensembles of Ginzburg-Landau type. For the sake of brevity, we do not carry out the generalization here.

Another type of generalization to consider is to ensembles of random holomorphic functions, for instance random holomorphic functions in the unit disc with various weighted norms or random entire functions on $\mathbb{C}$. The random analytic functions have an infinite number of zeros and one apparently needs to make a finite dimensional approximation to obtain a useful configuration space and a map to empirical measures.

An interesting question is whether one can generalize the large deviations results to higher dimensions. One could consider the hypersurface zero set of a single random section, or the joint zero set of a full system of m sections in dimension m. The rate function $\tilde{I}^{h,K}$ has a generalization to all dimensions, and so the large deviations result might admit a generalization. But the approach of this article, to extract the large deviations rate function from the joint probability current of zeros, does not seem to generalize well to higher dimensions. In dimension one, there exists a simple configuration space of possible zero sets of sections, but in higher dimensions there is no manageable analogue. It is possible that one can avoid this impasse by working on the space of potentials $\frac{1}{N}\log||s||_{h^N}$ rather than on the configuration

[1]In fact, an earlier posted version of the current article had the title *Large deviations of empirical zero point measures on Riemann surfaces, I: $g = 0$*

space of zeros. But it appears that one would have to extract the rate function directly from the potentials without using zeros coordinates. This circle of problems fits in very naturally with the Kähler potential theory of [GZ, BB, Ber1, Ber2], and it would be interesting to explore it further.

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2. Background

Polynomials of degree N on $\mathbb{C}$ may be viewed as meromorphic functions on $\mathbb{CP}^1$ with a pole of order N at ∞, or equivalently as holomorphic sections of the N power $\mathcal{O}(N) \to \mathbb{CP}^1$ of the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{CP}^1$, or again as homogeneous holomorphic polynomials of degree N on $\mathbb{C}^2$. It is useful to employ the geometric language of line bundles, Hermitian metrics and curvature, and in [Z] this language is indispensible. We briefly recall the relevant definitions, referring the reader to [GH] for further details.

We use the following standard notation: $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Also, $\partial f = \frac{\partial f}{\partial z} dz$ and similarly for $\bar{\partial} f$. The Euclidean Laplacian is given by $\Delta = 4\, \frac{\partial^2}{\partial z \partial \bar{z}}$ and $\partial\bar{\partial} = \frac{\partial^2}{\partial z \partial \bar{z}} dz \wedge d\bar{z}$. It is often convenient to use the real operators $d = \partial + \bar{\partial}, d^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$ and $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$. Thus, $dd^c f = \frac{i}{8\pi}\Delta f dz \wedge d\bar{z} = \frac{1}{4\pi}\Delta f dx \wedge dy$. We will often need the classical formula,

$$\Delta\left(\frac{1}{2\pi}\log|z|\right) = \delta_0 \iff dd^c(2\log|z|) = \delta_0 dx \wedge dy. \tag{26}$$

Henceforth, we regard δ_0 as a $(1,1)$ current so that δ_0 and $\delta_0 dx \wedge dy$ have the same meaning.

A smooth Hermitian metric h on a holomorphic line bundle L is a smooth family h_z of Hermitian inner products on the one-dimensional complex vector spaces L_z. Its Chern form is defined by

$$c_1(h) = \omega_h := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|e_L\|_h^2\,, \tag{27}$$

where e_L denotes a local holomorphic frame (= nonvanishing section) of L over an open set $U \subset M$, and $\|e_L\|_h = h(e_L, e_L)^{1/2}$ denotes the h-norm of e_L. We say that h is positive if the (real) 2-form ω_h is a positive $(1,1)$ form, i.e. defines a Kähler metric. For any smooth Hermitian metric h and local frame e_L for L, we write $\|e_L\|_h^2 = e^{-\varphi}$ (or, $h = e^{-\varphi}$), and

$$\omega_h = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi = dd^c\varphi.$$

We refer to $\varphi = -\log||e_L||_h^2$ as the potential of ω_h in U, or as the Kähler potential when ω_h is a Kähler form. We are interested in general smooth metrics, not only those where ω_h is positive; for instance, our methods and results apply in the case where $\varphi = 0$ (i.e. the metric is flat) on the support of $d\nu$. The metric h induces Hermitian metrics h^N on $L^N = L \otimes \cdots \otimes L$ given by $\|s^{\otimes N}\|_{h^N} = \|s\|_h^N$. The N-dependent factor $e^{-N\varphi}$ is then the local expression of h^N in the local frame e^N. We will only be considering the line bundles $\mathcal{O}(N) \to \mathbb{CP}^1$ in this paper.

We now specialize to the hyperplane line bundle $\mathcal{O}(1) \to \mathbb{CP}^1$ and its powers. We recall that $\mathbb{CP}^1$ is the set of lines through 0 in $\mathbb{C}^2$. The line through (z_0, z_1) is denoted $[z_0, z_1]$, which are the the homogeneous coordinates of the line. In the case of $\mathbb{CP}^1$ there exists a single holomorphic line bundle L^N of each degree. One writes $L = \mathcal{O}(1)$ and $L^N = \mathcal{O}(N)$. The bundle $\mathcal{O}(1)$ is dual to the tautological line bundle $\mathcal{O}(-1) \to \mathbb{CP}^1$ whose fiber at a point $[z_0, z_1] \in \mathbb{CP}^1$ is the line $[z_0, z_1]$ in $\mathbb{C}^2$. The line bundle $\mathcal{O}(1)$ is defined by two charts $U_1 = \mathbb{CP}^1 \backslash \{\infty\}$ $(z_0 \neq 0)$ and $U_2 = \mathbb{CP}^1 \backslash \{0\}$ $(z_1 \neq 0)$. A frame (nowhere vanishing holomorphic section) of $\mathcal{O}(-1)$ over U_1 is given by $e_1^*([z_0, z_1]) = (1, \frac{z_1}{z_0})$, and over U_2 by $e_2^*([z_0, z_1]) = (\frac{z_0}{z_1}, 1)$. The dual frames are the homogeneous polynomials on $\mathbb{C}^2$ defined by $e_1(z_0, z_1) = z_0$, resp. $e_2(z_0, z_1) = z_1$.

The potential φ is only defined relative to a frame, and we will need to know how it changes under a change of frame. Suppose that φ_1 is the potential of ω_h in the frame e_1, i.e. $||e_1([z_0, z_1])||_h^2 = e^{-\varphi_1}$. We assume that h, hence φ_1 is smooth in U_1 and we may (with a slight abuse of notation) regard it as a function on U_1 or on $\mathbb{C}$ in the standard coordinate $[z_0, z_1] \to \frac{z_0}{z_1} = w$. In the frame e_2 we have the local potential $||e_2([z_0, z_1])||_h^2 = e^{-\varphi_2}$ for some $\varphi_2 \in C^\infty(\mathbb{CP}^1 \backslash \{0\})$, which we identify with a function on $\mathbb{C}$. On the overlap $\mathbb{CP}^1 \backslash \{0, \infty\}$ the frames e_1, e_2 are related by $e_2([z_0, z_1]) = \frac{z_1}{z_0} e_1([z_0, z_1])$, so $||e_2([z_0, z_1])||_h^2 = |\frac{z_1}{z_0}|^2 ||e_1([z_0, z_1])||_h^2$. It follows that $\varphi_2([z_0, z_1]) = \varphi_1([z_0, z_1]) - 2 \log |\frac{z_1}{z_0}|$. If we use $w = \frac{z_0}{z_1}$ as a local coordinate, then $\varphi_2(w) = \varphi_1(\frac{1}{w}) + \log |w|^2$. As an illustration, the Kähler potential of the Fubini-Study metric on $\mathcal{O}(1)$ is given by $\log(1 + |w|^2) = \log(1 + \frac{1}{|w|^2}) + \log |w|^2$ in the two charts.

An important observation in understanding the global nature of (24) is the following:

LEMMA **5.** *The* $(1,1)$ *form* $e^{-2\varphi_1(z)} dz \wedge d\bar{z}$ *in the chart* U_1 *extends to a global smooth* $(1,1)$ *form* κ *on* $\mathbb{CP}^1$. *In the chart* U_2 *it equals* $e^{-2\varphi_2(z)} dz \wedge d\bar{z}$.

Proof. We need to check its invariance under the change of variables $\sigma(z) = \frac{1}{z}$. We have,

$$\sigma^* e^{-2\varphi_1(z)} dz \wedge d\bar{z} = e^{-2\varphi_1(\frac{1}{z})} \frac{dz \wedge d\bar{z}}{|z|^4}.$$

Since $\varphi_1(\frac{1}{z}) = \varphi_2(z) - \log |z|^2$, this is

$$e^{-2\varphi_2(z)} e^{2 \log |z|^2} \frac{dz \wedge d\bar{z}}{|z|^4} = e^{-2\varphi_2(z)} dz \wedge d\bar{z}.$$

□

2.1. **Poincaré-Lelong formula for the empirical measure of zeros.** The empirical measure of zeros Z_s (1) is given by (one-dimensional) Poincaré-Lelong formula,

$$\begin{aligned} Z_s = \tfrac{i}{\pi N} \partial\bar{\partial} \log |f| &= \tfrac{i}{N\pi} \partial\bar{\partial} \log \|s\|_{h^N} + \omega_h \\ &= \tfrac{2}{N} dd^c \log \|s\|_{h^N} + \omega_h \, . \end{aligned}$$

It is completely elementary in dimension one.

2.2. dd^c **Lemma.** We will need the dd^c Lemma on not-necessarily-positive $(1,1)$ currents. The dd^c Lemma on forms (cf. [Dem], Lemma 8.6 of Chapter VI) asserts that on a compact Kähler manifold, a d-closed (p,q) form u may be expressed as $u = dd^c v$ where v is a $(p -$

$1, q-1)$ form. The same Lemma is true for currents, with the change that v is only asserted to be a current.

When ω, ω' are two cohomologous positive closed $(1,1)$ currents (which on $\mathbb{CP}^1$ simply means $\int_{\mathbb{CP}^1} \omega = \int_{\mathbb{CP}^1} \omega'$), then one has a regularity theorem: $\omega - \omega' = dd^c\psi$ where $\psi \in L^1(\mathbb{CP}^1, \mathbb{R})$. We refer to [GZ], Proposition 1.4.

2.3. Hermitian inner products and Gaussian measures on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$.

We denote by $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ the space of holomorphic sections of $\mathcal{O}(N)$. It is well-known that they correspond to polynomials of degree N, which are their local expressions in the affine chart $U = \mathbb{CP}^1 \backslash \{\infty\}$ (see [GH]).

As mentioned in the introduction, the data (h, ν) determine inner products $G_N(h, \nu)$ on the complex vector spaces $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ (see (2) and (10)). An inner product on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$ induces a Gaussian measure on this complex vector space by the formula

$$d\gamma_N(s_N) := \frac{1}{\pi^{d_N}} e^{-|c|^2} dc\,, \quad s_N = \sum_{j=1}^{d_N} c_j S_j^N\,, \quad c = (c_1, \dots, c_{d_N}) \in \mathbb{C}^{d_N}\,,$$

where $d_N = N+1$, $\{S_1^N, \dots, S_{d_N}^N\}$ is an orthonormal basis for $H^0(\mathbb{CP}^1, \mathcal{O}(N))$, and dc denotes $2d_N$-dimensional Lebesgue measure. The measure γ_N is characterized by the property that the $2d_N$ real variables $\Re c_j, \Im c_j$ $(j = 1, \dots, d_N)$ are independent Gaussian random variables with mean 0 and variance 1/2; equivalently,

$$\mathbf{E}_N c_j = 0, \quad \mathbf{E}_N c_j c_k = 0, \quad \mathbf{E}_N c_j \bar{c}_k = \delta_{jk}\,,$$

where $\mathbf{E}_N$ denotes the expectation with respect to the measure γ_N.

In §3, we will define an essentially equivalent Fubini-Study volume form on the projective space of sections $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$.

3. Joint probability current of zeros and the Fubini-Study volume form

In this section, we define the principal object of this article, the joint probability current of zeros. We then prove the first part (23) of Proposition 3, giving the formula for the joint probability current of zeros as the pull back to configuration space of the Fubini-Study volume form on the projective space of sections. We rewrite the formula in terms of the Green's function in the next section.

3.1. The joint probability current of zeros.

The joint probability current of zeros is defined by

$$\vec{K}_N^N(z^1, \dots, z^N) := \mathbf{E}(Z_s(z^1) \otimes Z_s(z^2) \otimes \cdots \otimes Z_s(z^N)).$$

It is a current on the configuration space $(\mathbb{CP}^1)^{(N)}$ of N points. It is the extreme case $n = N$ of the n-point zero correlation current

$$\vec{K}_n^N(z^1, \dots, z^n) := \mathbf{E}(Z_s(z^1) \otimes Z_s(z^2) \otimes \cdots \otimes Z_s(z^n)) \tag{28}$$

on the configuration space $(\mathbb{CP}^1)^{(n)}$. Recall that by a current we mean a linear functional on test forms, i.e. for any test function $\varphi_1(z^1) \otimes \cdots \otimes \varphi_n(z^n) \in C((\mathbb{CP}^1)^{(n)}$,

$$\left(\vec{K}_n^N(z^1, \dots, z^n), \varphi_1(z^1) \otimes \cdots \otimes \varphi_n(z^n)\right) = \mathbf{E}\left[\left(Z_s, \varphi_1\right)\left(Z_s, \varphi_2\right) \cdots \left(Z_s, \varphi_n\right)\right].$$

3.2. **Fubini-Study formula.** We now present the most useful approach to the joint probability current of zeros in the case of genus zero.

It is a classical fact that the projective space of sections $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ may be identified with the configuration space $(\mathbb{CP}^1)^{(N)}$ of N points of $\mathbb{CP}^1$. This essentially comes down to the elementary fact that a set $\{\zeta_1, \dots, \zeta_N\}$ determines a line of polynomials $[P_\zeta] \in \mathbb{P}\mathcal{P}_N$ of degree N, at least when none of the zeros occur at ∞. Viewed as holomorphic sections of $\mathcal{O}(N) \to \mathbb{CP}^1$ one can also allow ∞ to be a zero and then N points of $\mathbb{CP}^1$ corresponds to a line of holomorphic sections.

The correspondence $\zeta \to [P_\zeta]$ defines a line bundle

$$\mathcal{Z}_N \to (\mathbb{CP}^1)^{(N)}, \;\; (\mathcal{Z}_N)_\zeta = \{[p] \in \mathcal{P}_N : \mathcal{D}(p) = \zeta\}, \tag{29}$$

i.e. the fiber of $\mathcal{Z}_N$ at $\zeta_1 + \cdots + \zeta_N$ is the line $\mathbb{C}P_\zeta$ of holomorphic sections of $\mathcal{O}(N)$ with the divisor $\zeta = \zeta_1 + \cdots + \zeta_N$. It is isomorphic to the bundle $\mathcal{O}(1) \to \mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ under the identification $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N)) = (\mathbb{CP}^1)^{(N)}$. One can construct a form representing the first Chern class $c_1(\mathcal{Z}_N)$ using a Hermitian inner product on $\mathcal{Z}_N$ or equivalently a Hermitian inner product on $H^0(\mathbb{CP}^1, \mathcal{O}(N))$: at a point $\zeta \in (\mathbb{CP}^1)^{(N)}$, the G_z-norm of a vector $P_\zeta \in \mathcal{Z}_\zeta$ is $||P_\zeta||_G$, the norm of P_ζ as an element of $H^0(\mathbb{CP}^1, \mathcal{O}(N))$. This is the Fubini-Study Hermitian metric determined by the inner product.

Let us recall the basic definitions and formulae in the case of the standard inner product on $\mathbb{C}^{d+1}$ and $\mathbb{CP}^d$. Let $Z \in \mathbb{C}^{d+1}$ and let $||Z||^2 = \sum_{j=0}^d |Z_j|^2$. In the open dense chart $Z_0 \neq 0$, and in affine coordinates $w_j = \frac{Z_j}{Z_0}$, the Fubini-Study volume form is given by,

$$d\mathrm{Vol}_I = \frac{\prod_i dw_i \wedge d\bar{w}_i}{(1 + ||w||^2)^{d+1}}.$$

For our purposes, it is more useful to lift this form to $\mathbb{C}^{d+1}$ under the natural projection, $\pi : \mathbb{C}^{d+1} - \{0\} \to \mathbb{CP}^d$. A straightforward calculation shows that

$$\pi^* d\mathrm{Vol}_I = ||Z_0||^2 \frac{\prod_{j=1}^d dZ_j \wedge d\bar{Z}_j}{||Z||^{2(d+1)}},$$

in the sense that

$$\frac{dZ_0 \wedge d\bar{Z}_0}{|Z_0|^2} \wedge \frac{\prod_{i=1}^d dw_i \wedge d\bar{w}_i}{(1 + ||w||^2)^{d+1}} = \frac{\prod_{j=0}^d dZ_j \wedge d\bar{Z}_j}{||Z||^{2(d+1)}}.$$

We need a more general formula where the inner product $||Z||^2$ is replaced by any Hermitian inner product on $\mathbb{C}^{d+1}$. We recall that the space of Hermitian inner products on $\mathbb{C}^{d+1}$ is the symmetric space $GL(d+1, \mathbb{C})/U(d+1)$. If we fix the standard inner product (v, w), then any other inner product has the form $G(v, w) = (Pv, w)$ where P is a positive Hermitian matrix. It has the form $P = A^* A$ where $A \in GL(d+1, \mathbb{C})$.

Suppose, then, that instead of the standard inner norm $||Z||$ on $\mathbb{C}^{d+1}$ we are given the norm $||AZ||$ where $A \in GL(d+1, \mathbb{C})$. Then the Fubini-Study metric becomes $\partial\bar{\partial} \log ||AZ||^2$. Since the linear transformation defined by A is holomorphic, the associated volume form

dV_A is simply the pull-back by A of the previous form,

$$\begin{aligned}\pi^* d\mathrm{Vol}_A &= A^* \frac{(\partial\bar\partial \log ||Z||^2)^{d+1}}{\frac{dZ_0 \wedge d\bar Z_0}{|Z_0|^2}} = \frac{|(AZ)_0|^2}{||AZ||^{2(d+1)}} A^* \left(\frac{\prod_{j=0}^d dZ_j \wedge d\bar Z_j}{dZ_0 \wedge d\bar Z_0} \right) \\ &= |\det A|^2 |(AZ)_0|^2 \cdot \left(\frac{\partial}{\partial Z_0} \wedge \frac{\partial}{\partial \bar Z_0} \vdash A^*(dZ_0 \wedge d\bar Z_0) \right)^{-1} \left(\frac{\prod_{j=1}^d dZ_j \wedge d\bar Z_j}{||AZ||^{2(d+1)}} \right).\end{aligned} \tag{30}$$

Here, $\left(\frac{\partial}{\partial Z_0} \wedge \frac{\partial}{\partial \bar Z_0} \vdash A^*(dZ_0 \wedge d\bar Z_0)\right)$ is the coefficient of $dZ_0 \wedge d\bar Z_0$ in the form $d(A^*Z)_0 \wedge \overline{d(A^*Z)_0}$.

We now prove the first part of Proposition 3.

3.3. **Proof of** (23) **in Proposition 3.** To prove (23), we use (30) and change variables to zeros coordinates.

We first consider the change of variables in local coordinates on $\mathbb{CP}^1$. We fix the usual affine chart $U \subset \mathbb{C}$ and let z be the local coordinate. We then have a corresponding local coordinate system $(\zeta_1, \ldots, \zeta_N)$ on $(\mathbb{CP}^1)^N$ which is defined in the chart $(\mathbb{C})^N$.

We have defined the joint probability current (28) as an (N, N) form on configuration space $(\mathbb{CP}^1)^N$. It pulls back under the S_N cover $(\mathbb{CP}^1)^N \to (\mathbb{CP}^1)^{(N)}$ and we wish to express it in the local coordinate system $(\zeta_1, \ldots, \zeta_N)$ to obtain the formula in Proposition 3. We then write down its density with respect to the local Lebesgue volume form $d^2\zeta_1 \cdots d^2\zeta_N$ of the chart.

To prove the Proposition, we start with the Newton-Vieta's formula:

$$\prod_{j=1}^N (z - \zeta_j) = \sum_{k=0}^N (-1)^k e_{N-k}(\zeta_1, \ldots, \zeta_N) \; z^k. \tag{31}$$

Here, the elementary symmetric functions are defined by

$$e_j = \sum_{1 \le p_1 < \cdots < p_j \le N} z_{p_1} \cdots z_{p_j}.$$

As mentioned above, the formula (31) defines a map $(\mathbb{CP}^1)^{(N)} \to \mathcal{P}_N$, which is a section of the line bundle $\mathcal{Z}_N$ (29). It is the section taking its values in the polynomials $\sum_{i=0}^N a_i z^i$ for which $a_N = 1$. Since $e_0(\zeta) \equiv 1$, the linear coordinates are affine coordinates in the chart $c_0 = 1$, where c_j are coordinates with respect to the basis $\{z^j\}$. We then change variables from the Lebesgue volume form $da_1 \wedge d\bar a_1 \wedge \cdots \wedge da_N \wedge d\bar a_N$ in the affine chart to a volume form in the coordinates $(\zeta_1, \ldots, \zeta_N)$. It is well-known (see e.g. [LP]) that this change of variables has Jacobian $|\Delta(\zeta)|^2$ where as above, $\Delta(\zeta_1, \ldots, \zeta_N) = \prod_{1 \le j < k \le N} (\zeta_k - \zeta_j)$ is the Vandermonde determinant.

We now express the Fubini-Study probability measure on $\mathbb{P}H^0(\mathbb{CP}^1, \mathcal{O}(N))$ in the coordinates ζ_j. The first problem we face is that the right side of (31) expresses the polynomial on the left side in coordinates with respect to the basis $\{z^j\}_{j=0}^N$, which is usually not an orthonormal basis with respect to the inner product (2). We need to make the additional change of variables from coordinates $\mathcal{E}_j$ with respect to an orthonormal basis $\{\psi_j\}$ for our inner product $G_N(h, \nu)$, $\prod_{j=1}^N (z - \zeta_j) = \sum_{\ell=0}^N \mathcal{E}_{N-\ell} \psi_\ell$ to coordinates $Z_j = (-1)^{N-j} e_{N-j}$ with respect to the monomial basis $\{z^j\}$. With no loss of generality, we assume that the orthogonal polynomials $\{\psi_j\}$ are enumerated according to degree, so that ψ_N is the unique polyomial in the basis with a z^N term. The change of basis matrix $\mathcal{A}_N(h,\nu)Z = \mathcal{E}$ is given by,

$$\left(\mathcal{A}_N^{jk}\right)_{j,k=0}^N = \left(\langle z^j, \psi_k\rangle_{G_N(h,\nu)}\right)_{j,k=0}^N. \tag{32}$$

Next we observe that

$$\frac{\partial}{\partial Z_0} \wedge \frac{\partial}{\partial \bar{Z}_0} \vdash \mathcal{A}_N(h,\nu)^*(dZ_0 \wedge d\bar{Z}_0) = |\mathcal{A}_N^{00}|^2.$$

Indeed, $\mathcal{A}_N^* dZ_0 = \sum_j \mathcal{A}_N^{0j} dZ_j$ and the desired expression is the coefficient of $dZ_0 \wedge d\bar{Z}_0$ in $d(\mathcal{A}_N^* Z)_0 \wedge d\overline{(\mathcal{A}_N^* Z)_0}$. We further observe that $|\mathcal{A}_N^{00}|^2$ is a constant independent of ζ. By our ordering, $\psi_N = k_N z^N + k_{N-1} z^{N-1} \cdots$ for some $k_N \neq 0$. Since

$$\prod_j (z - \zeta_j) = \sum_j \mathcal{E}_{N-j}\psi_j = z^N + e_1(\zeta) z^{N-1} + \cdots,$$

it follows that

$$\mathcal{A}_N^{00} = k_N^{-1}, \text{ and that } |(\mathcal{A}_N(h,\nu)Z)_0|^2 = k_N^{-2}.$$

Combining this evaluation with (30), we see that the pull back of the Fubini-Study volume form with respect to $G_N(h,\nu)$ to $\mathbb{C}^{N+1}$ is given by

$$|\det \mathcal{A}_N(h,\nu)|^2 \left(\frac{\prod_{j=1}^N dZ_j \wedge d\bar{Z}_j}{||\mathcal{A}_N(h,\nu)Z||^{2(N+1)}} \right). \tag{33}$$

We now change variables to zeros coordinates. As mentioned above, $\prod_{j=1}^N dZ_j \wedge d\bar{Z}_j = |\Delta(\zeta)|^2 \prod_j d^2\zeta_j$. The denominator in (33) equals the sum of the squares of the components of $\mathcal{A}_N(h,\nu)Z$, which is the L^2 norm-squared of $\prod_{j=1}^N (z-\zeta_j)$ with respect to $G_N(h,\nu)$, i.e.

$$||\mathcal{A}_N(h,\nu)Z||^{2(N+1)} = \left(\int_{\mathbb{CP}^1} \prod_{j=1}^N |(z-\zeta_j)|^2 e^{-N\varphi} d\nu(z) \right)^{N+1}.$$

Further,

$$\begin{aligned} |(\mathcal{A}_N(h,\nu)Z)_0|^2 = |\mathcal{E}_0(\zeta)|^2 &= |\langle \textstyle\prod_{j=1}^N (z-\zeta_j), \psi_N\rangle|^2 \\ &= \left| \textstyle\int_{\mathbb{C}} \prod_{j=1}^N (z-\zeta_j) \overline{\psi_N(z)} e^{-N\varphi(z)} d\nu(z) \right|^2. \end{aligned}$$

This completes the proof of (23). □

We refer to the coefficient of $d^2\zeta_1 \cdots d^2\zeta_N$ in (23) as the joint probability density (JPD) of zeros:

$$D_N(\zeta_1,\ldots,\zeta_N) = |\det \mathcal{A}_N(h,\nu)|^2 \frac{|\Delta(\zeta_1,\ldots,\zeta_N)|^2}{\left(\int_{\mathbb{CP}^1} \prod_{j=1}^N |(z-\zeta_j)|^2 e^{-N\varphi} d\nu(z) \right)^{N+1}}. \tag{34}$$

Remark: The elementary symmetric functions $e_j(\zeta)$ of $\zeta = (\zeta_1,\ldots,\zeta_N)$ are natural coordinates in $\mathbb{C}^{(N)}$, and a natural holomorphic volume form is given by

$$\Omega_{C^{(N)}} = de_1 \wedge \cdots \wedge de_N \tag{35}$$

while the corresponding (N, N) form is

$$\Omega_{C^{(N)}} \wedge \overline{\Omega}_{C^{(N)}} = de_1 \wedge d\bar{e_1} \wedge \cdots \wedge de_N \wedge d\bar{e_N} = |\Delta(\zeta_1, \ldots, \zeta_N)|^2 d^2\zeta_1 \cdots d^2\zeta_N.$$

3.4. **Intrinsic formula for the joint probability current.** The Fubini-Study form has an intrinsic geometric interpretation as the curvature form (27) for the Hermitian line bundle $\mathcal{Z}_N \to (\mathbb{CP}^1)^{(N)}$ equipped with its metric $G(h^N, \nu)$. This is of independent geometric interest and we pause to consider it.

A local frame for $\mathcal{Z}_N$ (henceforth we drop the N for notation simplicity) is a non-vanishing holomorphic selection of a polynomial P_ζ from the line $\mathbb{C}P_\zeta$ of polynomials (or more generally, holomorphic sections of $\mathcal{O}(N) \to \mathbb{CP}^1$) with divisor ζ. The standard choice is to trivialize $\mathcal{Z}$ over $(\mathbb{C})^N$ using the section $P_\zeta(z) = \prod_{j=1}^N (z - \zeta_j) e^N(z)$ where $e(z)$ is the standard affine frame of $\mathcal{O}(1) \to \mathbb{CP}^1$ over $\mathbb{C}$. In this article, the inner product $G = G_N(h, \nu)$ is defined by (10). It follows that the curvature $(1,1)$ form of $\mathcal{Z}$ is given by

$$\omega_{\mathcal{Z}} = \frac{i}{2} \partial\bar{\partial} \log ||P_\zeta||_{G(h,\nu)}, \tag{36}$$

where $\partial\bar{\partial}$ is the operator on $(\mathbb{CP}^1)^{(N)}$. Thus,

$$\Phi_N(\zeta) := \log ||P_\zeta||_{G(h,\nu)}$$

is the Kähler potential for the Kähler form of configuration space, and the volume form is given by

$$dV_{FS, G_N(h,\nu)} = \left(\frac{i}{2} \partial\bar{\partial} \log ||P_\zeta||_{G_N(h,\nu)} \right)^N,$$

the (N, N) form defined as the top exterior power of (36). What (23) asserts is thus equivalent to

Proposition **6.** *We have,*

$$\left(\frac{i}{2} \partial\bar{\partial} \Phi_N \right)^N = |\det \mathcal{A}_N(h, \nu)|^2 |\Delta(\zeta)|^2 e^{-(N+1)\Phi_N(\zeta)} \Pi_{j=1}^N d^2\zeta_j.$$

This Proposition clarifies in what sense the right hand side is a well-defined volume form on $(\mathbb{CP}^1)^{(N)}$. Namely, it corresponds to the choice of the Kähler potential Φ_N, i.e. the expression of the Hermitian metric G on $\mathcal{Z}$ in the local frame P_ζ.

4. Green's functions and the joint probability current: completion of the proof of Proposition 3

As discussed in the introduction, it is very helpful to express the joint probability current and rate function in terms of global objects on $\mathbb{CP}^1$. In the statement of Theorem 1, we expressed $I^{h,K}$ in terms of the Green's function G_h. In this section, we give background on the definition and properties of Green's function that are needed in the proof of Theorem 1. The main result is Proposition 17, in which we express the joint probability current in terms of Green's functions, and thus complete the proof of Proposition 3.

4.1. **Green's function for ω_h.** The Green's function $G_h(z, w)$ is defined in (5). We now verify that G_h is well-defined, that it is smooth outside of the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and that its only singularity is a logarithmic singularity on the diagonal. We sometimes write $g_z(w) = G(z, w)$ to emphasize that the derivatives in (5) are in the w variable. When ω_h is a Kähler metric, $g_z(w)$ is a special case of the notion of Green's current for the divisor $\{z\}$. For background we refer to [He], although it only discusses the case where ω_h is a Kähler form. We also refer to [ABMNV] for background on global analysis on Riemann surfaces.

When we express the Green's function in the charts $U_1 \times U_1$, resp. $U_2 \times U_2$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$, we subscript G_h accordingly. We also drop the subscript h for simplicity of notation when the metric is understood.

PROPOSITION **7.** *There exists a unique function $G_h(z, w) \in L^1(\mathbb{CP}^1 \times \mathbb{CP}^1)$ solving the system of equations (5). When $z \neq \infty$, in the local affine chart $\mathbb{C}$ it is given by (6). Under the holomorphic map $z \to \frac{1}{z}$, we have*

$$G_1(\frac{1}{z}, \frac{1}{w}) = G_2(z, w).$$

Proof. Given any $z \in \mathbb{CP}^1$, there exists a section $s_z \in H^0(\mathbb{CP}^1, \mathcal{O}(1))$ which vanishes at z. There exists a distinguished section (denoted $\mathbf{1}_z(w)$ in [ABMNV]) which has the Taylor expansion $w - z$ in the standard affine frame and which corresponds to the meromorphic function $w - z$. When $z = \infty$, $s_\infty(w)$ corresponds to the meromorphic function 1. As a homogeneous polynomial of degree one in each variable on $\mathbb{C}^2 \times \mathbb{C}^2$ it is given by $w_1 z_0 - z_1 w_0$. We view the two-variable section $s_w(z)$ as a section of $\pi_1^* \mathcal{O}(1) \boxtimes \pi_2^* \mathcal{O}(1) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ and equip the line bundle with the product Hermitian metric $h_z \boxtimes h_w$ (here and in what follows, $\boxtimes$ denotes the exterior tensor product on $\mathbb{CP}^1 \times \mathbb{CP}^1$). We then claim that (with $E(h)$ defined in (7)),

$$G_h(z, w) = \log ||s_z(w)||^2_{h_z \boxtimes h_w} - E(h)$$

satisfies (i)-(iii) of (5) for all z. Both (i) and (ii) are clear from the formula and from (2.1).

To prove (iii) and the identity claimed in the Proposition, it is convenient to use the local affine frames e_j of $\mathcal{O}(1) \to \mathbb{CP}^1$ over the affine charts U_j (see §2 for notation).

LEMMA **8.** *There exists a constant $E(h)$ so that, in the affine chart U_j $(j = 1, 2)$ and all $z \in \mathbb{C}$,*

$$G_j(z, w) = 2 \log |z - w| - \varphi_j(z) - \varphi_j(w) + E(h),$$

and $\int_{\mathbb{C}} G_j(z, w) dd^c \varphi_j = 0$.

Indeed, in U_1 we put $z_0 = w_0 = 1$ and $z_1 = z, w_1 = w$, and then

$$\log ||s_z(w)||^2_{h_z \boxtimes h_w} = 2 \log |z - w| - \varphi_1(z) - \varphi_1(w). \tag{37}$$

In U_2 we put $z_1 = w_1 = 1$ and $z_0 = z, w_0 = w$ and obtain the same expression with φ_2 replacing φ_1. On the overlap, the stated identity follows from the fact that

$$2 \log |\frac{1}{z} - \frac{1}{w}| - \varphi_1(\frac{1}{z}) - \varphi_1(\frac{1}{w}) = 2 \log |z - w| - \varphi_1(\frac{1}{z}) - \varphi_1(\frac{1}{w}) - 2 \log |z| - 2 \log |w|,$$

and the fact that $\varphi_2(w) = \varphi_1(\frac{1}{w}) + \log |w|^2$ (see §2).

To complete the proof, we need to show that $\int_{\mathbb{CP}^1} \log ||z-w||^2_{h_z \boxtimes h_w} \omega_h$ is a constant in z. In fact we claim that when $z, w \in U_1$, then

$$\int_{\mathbb{CP}^1} \log ||z-w||^2_{h_z \boxtimes h_w} \omega_h = -\int \varphi \omega_h - 4\pi \rho_\varphi(\infty).$$

The calculation of this integral can be done by the integration by parts formulae in §8.2. We use (37) to break up the integrand into three terms. The second integrates to $-\varphi(z) \int_{\mathbb{CP}^1} \omega_h = -\varphi(z)$, while the third integrates to $-\int \varphi \omega_h = -\int \varphi dd^c \varphi$. The first (logarithmic) term is an integral of the type studied in (1) of Lemma 34, where it is evaluated as

$$\int_{\mathbb{C}} 2 \log |z-w| dd^c \varphi_1 = \varphi_1(z) - 4\pi \rho_{\varphi_1}(\infty). \tag{38}$$

In the full sum, the $\varphi_1(z)$ terms cancel, leaving the stated expression. The same integral holds with φ_2 replaced by φ_1 if $z, w \in U_2$ by the identity in the Proposition. This proves the integral formula in all cases. □

As an example of the calculation, the Fubini-Study Green's function is given in the chart $U_1 \times U_1$ by $G_{FS}(z,w) = 2\log[z,w]^2 - C$, where $[z,w] = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$. The constant C is determined by the condition (iii). To study its behavior when $z = \infty$ we change coordinates $\sigma : z \to \frac{1}{z}, w \to \frac{1}{w}$ and study the behavior at 0. The distance $[z,w]$ and Green's function are invariant under the isometry σ, so we obtain the same expression after the change of coordinates. In particular, in these coordinates, $G_{FS}(\infty, u) = 2\log|u| - \log(1+|u|^2) = \varphi(\frac{1}{u})$, where $\varphi_{FS}(w) = \log(1+|w|^2)$.

Remark: We note that a local Kähler potential φ (or a global relative Kähler potential) is only unique up to an additive constant. One may normalize φ by the condition $\int_{\mathbb{CP}^1} \varphi \omega_h = 0$. However, in the above formula we have not done so. We observe that the Green's function is (as it must be) invariant under addition of a constant to φ.

4.2. **Green's potential of a measure.** We now return to the Green's potential (8) and Green's energy (9) of the introduction. Given a real $(1,1)$ form ω on $\mathbb{CP}^1$, we define

$$SH(\mathbb{CP}^1, \omega) := \{u \in L^1(\mathbb{CP}^1, \mathbb{R} \cup \{-\infty\}) : dd^c u + \omega \geq 0\}. \tag{39}$$

For any closed $(1,1)$ form, the $\partial\bar{\partial}$ Lemma implies that the map

$$\psi \to \omega_\psi := \omega + dd^c \psi \in \mathcal{M}(\mathbb{CP}^1) \tag{40}$$

is surjective and has only constants in its kernel, i.e.

$$SH(\mathbb{CP}^1, \omega) \simeq \mathcal{M}(\mathbb{CP}^1) \oplus \mathbb{R}.$$

The Green's potential (compare with (8)) of a measure defines a global inverse to (40) and is uniquely characterized as the solution of

$$\begin{cases} dd^c U^\mu_\omega = \mu - \omega; \\ \int_{\mathbb{CP}^1} U^\mu_\omega \omega = 0. \end{cases}$$

Any smooth integral $(1,1)$ form $\omega \in H^2(\mathbb{CP}^1, \mathbb{Z})$ is the curvature $(1,1)$ form of a smooth Hermitian metric h (see §2), and we subscript the potential by h rather than ω. Thus,

$$dd^c U_h^\mu(z) = \mu - \omega_h. \tag{41}$$

We illustrate Green's potentials in the important case where $\mu = \mu_\zeta$. In Lemma 15, we will essentially write the ω_h-subharmonic function $\frac{1}{N} \log ||s_\zeta(z)||_{h^N}$ with

$$s_\zeta(z) = \prod_{j=1}^N (z - \zeta_j) e^N(z) \tag{42}$$

as a Green's potential. To tie the discussions together, we note that the special case $\omega = \omega_h$ of Lemma 15 below can be reformulated in terms of Green's potentials as follows:

LEMMA **9.** *We have,*

- *(i)* $\frac{1}{N} \log ||s_\zeta(z)||_{h^N} - \frac{1}{N} \int_{\mathbb{CP}^1} \log ||s_\zeta||^2_{h^N} \omega_h = U_h^{\mu_\zeta}(z)$. *Hence,*

  $||s_\zeta(z)||_{h^N}^{\frac{1}{N}} e^{-\frac{1}{N} \int_{\mathbb{CP}^1} \log ||s_\zeta||^2_{h^N} \omega_h} = e^{U_h^{\mu_\zeta}}$.

- *(ii)* $\int \log ||s_\zeta(w)||^2_{h^N} \omega_h = \int_{\mathbb{C}} \log ||s_\zeta(w)||^2_{h^N} dd^c \varphi = N(\int \varphi d\mu_\zeta - E(h))$.

Proof. Since $d\mu_\zeta = dd^c \frac{1}{N} \log ||s_\zeta(z)||^2_{h^N} + \omega_h$,

$$\begin{aligned} U_h^{\mu_\zeta}(z) &:= \int_{\mathbb{CP}^1} G_h(z,w) d\mu_\zeta(w) \\ &= \int_{\mathbb{CP}^1} G_h(z,w) (\tfrac{1}{N} dd^c \log ||s_\zeta(w)||^2_{h^N} + \omega_h) \\ &= \int_{\mathbb{CP}^1} G_h(z,w) \tfrac{1}{N} dd^c \log ||s_\zeta(w)||^2_{h^N} \\ &= \tfrac{1}{N} \log ||s_\zeta||^2_{h^N}(z) - \tfrac{1}{N} \int_{\mathbb{CP}^1} \log ||s_\zeta||^2_{h^N}(z) \omega_h. \end{aligned}$$

This proves the first point (i) of the lemma. The proof of (ii) is given in Lemma 34 in the Appendix. It is proved by integration by parts (see (64) of §8 for the required constant coming from the boundary term "at infinity") together with the Poincaré-Lelong formula (2.1). □

4.3. **Regularity of Green's functions.** For use in the proof of the large deviation principle, we need the following regularity result on the Green's function. In what follows, $D = \{(z,z) : z \in \mathbb{CP}^1\}$.

PROPOSITION **10.** $G_h(z,w) \in C^\infty(\mathbb{CP}^1 \times \mathbb{CP}^1 \backslash D)$, *and in any local chart, near the diagonal it possesses the singularity expansion,*

$$G_h(z,w) = 2 \log |z - w| + \rho(z) + O(|z - w|)$$

where $\rho(z)$ *is a smooth function on* $\mathbb{CP}^1$ *known as the Robin constant. In particular,* $G_h(z, \cdot) \in L^1(\mathbb{CP}^1, \omega_h)$ *for any* z, *and there exists a constant* $C_G < \infty$ *so that*

$$\sup_{(z,w) \in \mathbb{CP}^1 \times \mathbb{CP}^1} G(z,w) \leq C_G.$$

Proof. When ω is a Kähler metric, we may form its Laplacian Δ_ω and then the Green's function $G_\omega(z,w)$ is the kernel of Δ_ω^{-1} on the orthogonal complement of the constant functions. Thus, in the compact case, G_ω is defined by two conditions:

(1) $\Delta_\omega G_\omega(z,w) = \delta_z(w) - \frac{1}{A}$, where $A = \int_{\mathbb{CP}^1} \omega$. That is, $G_\omega(z,w)$ is a (singular) ω-subharmonic function. In our case $A = 1$.
(2) $\int_{\mathbb{CP}^1} G_\omega(z,w)\omega = 0$.

We denote by $\{\varphi_j\}_{j=0}^\infty$ an orthonormal basis of eigenfunctions of Δ_ω in $L^2(\mathbb{CP}^1,\omega)$, with $\varphi_0 = \frac{1}{\sqrt{A}}$ and with $\Delta\varphi_j = \lambda_j\varphi_j$ with $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \downarrow -\infty$. Then G_ω has the eigenfunction expansion,

$$G_\omega(z,w) = \sum_{j=1}^\infty \frac{\varphi_j(z)\varphi_j(w)}{\lambda_j}. \tag{43}$$

The singularity expansion near the diagonal is then a standard fact which follows from the Hadamard-Riesz parametrix method (see [HoIII], Section 17.4).

We now consider general smooth $(1,1)$ form ω_h. When ω_h fails to be Kähler , we introduce a Kähler metric ω in the same cohomology class as ω_h. Since $\int_{\mathbb{CP}^1}\omega = \int_{\mathbb{CP}^1}\omega_h$, the $\partial\bar\partial$ Lemma implies that there exists a relative Kähler potential φ_{hg} such that $\omega_h - \omega = dd^c\varphi_{hg}$. By definition (8), the relative potentials are given by

$$dd^c U_h^\omega = \omega - \omega_h, \quad dd^c U_\omega^{\omega_h} = \omega_h - \omega. \tag{44}$$

It follows that $U_h^\omega = -U_\omega^{\omega_h} + A_{gh}$ for a constant A_{gh}, and from $\int U_\omega^{\omega_h}\omega = 0$ we have

$$U_h^\omega = -U_\omega^{\omega_h} + \int U_\omega^{\omega_h}\omega_h.$$

By integrating both sides against ω_h we also have $\int U_h^\omega \omega = \int U_\omega^{\omega_h}\omega_h$.

We then claim that

$$G_h(z,w) - G_\omega(z,w) = U_h^\omega(z) + U_h^\omega(w) - \int U_h^\omega \omega. \tag{45}$$

Since the relative potential U_h^ω is a solution of the elliptic equation (44), and the left side is C^∞, it follows that $U_h^\omega \in C^\infty$. Hence (45) implies the regularity result for any smooth h.

To conclude the proof, we need to prove the identity (45). We observe that the dd^c derivatives of both sides of (45) in either z or w agree, both equaling $\omega - \omega_h$. Hence, there exists a unique constant C_{gh} such that

$$G_h(z,w) - G_\omega(z,w) = U_h^\omega(z) + U_h^\omega(w) + C_{gh}. \tag{46}$$

To determine C_{gh} we integrate both sides of (46) against $\omega_h(z) \boxtimes \omega(w)$ and use that $\int G_\omega \omega = 0 = \int G_h \omega_h$. Hence,

$$C_{gh} = -(\int U_h^\omega \omega_h + \int U_h^\omega \omega) = -\int U_h^\omega \omega,$$

since $\int U_h^\omega \omega_h = 0$. This implies (45). ∎

Corollary **11.** *With C_G as in Proposition 10, for any $\mu \in \mathcal{M}(\mathbb{CP}^1)$, $\sup_z U_h^\mu \leq C_G$*

Proof. This follows from the fact that $U_h^\mu(z) = \int_{\mathbb{CP}^1} G_h(z,w)d\mu(w) \leq C_G$, as $\int d\mu = 1$. ∎

4.4. **Green's energy.** From (9), we have for the Green's energy with respect to ω_h

$$\mathcal{E}_h(\mu) = \int_{\mathbb{CP}^1} U_h^\mu(z) d\mu(z) = \int_{\mathbb{CP}^1} U_h^\mu(z)(dd^c U_h^\mu + \omega_h) = \int_{\mathbb{CP}^1} U_h^\mu(z) dd^c U_h^\mu,$$

where we used (41) in the second equality and the fact that $\int U_h^\mu \omega_h = 0$ in the last equation.

In the next result, we outline a proof of the convexity of the energy functional for general smooth Hermitian metrics. It is used in the proof of the convexity of the rate function in Lemma 28. Convexity of the energy is well-known in weighted potential theory: It is proved in Lemma 1.8 of [ST] that $-\Sigma(\mu) \geq 0$ in the case where case where $\mu = \mu_1 - \mu_2$ is a signed Borel measure with compact support, where $\mu(\mathbb{C}) = 0$ and each of μ_1, μ_2 satisfies $-\Sigma(\mu_j) < \infty$. A different proof is given in [BG], Property 2.1(4) and another in Proposition 5.5 of [BB]. We give a somewhat different proof in our setting of $\mathbb{CP}^1$.

We define the energy form on $\mathcal{M}(\mathbb{CP}^1)$ by

$$\langle \mu, \nu \rangle_\omega := \int_{\mathbb{CP}^1} G_\omega(z, w) d\mu(z) d\nu(w) = \int_{\mathbb{CP}^1} U_\omega^\mu d\nu = \int_{\mathbb{CP}^1} U_\omega^\nu d\mu. \tag{47}$$

As in [C], we denote the probability measures of finite energy $||\mu||_\omega^2 < \infty$ by $\mathcal{E}^+(\mathbb{CP}^1) \subset \mathcal{M}(\mathbb{CP}^1)$.

PROPOSITION **12.** *For any smooth Hermitian metric on $\mathcal{O}(1)$, $-\mathcal{E}_h$ is a strictly convex functional on $\mathcal{M}(\mathbb{CP}^1)$.*

Proof. We first prove strict convexity when ω is a Kähler metric.

LEMMA **13.** *When ω is a Kähler metric, the energy form $\langle \mu, \nu \rangle_\omega$ is negative semi-definite on signed measures of finite energy. The unique measures of energy zero are multiples of ω.*

Proof. From the eigenfunction expansion (43), it follows that

$$\langle \mu, \nu \rangle_\omega = \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_\omega(z, w) d\mu(z) d\nu(z) = \sum_{j=1}^\infty \frac{\mu(\varphi_j)\nu(\varphi_j)}{\lambda_j}.$$

It is clear that for any signed measure μ, $\langle \mu, \mu \rangle_\omega \leq 0$ with equality if and only if $\mu(\varphi_j) = 0$ for all $j = 1, 2, \ldots$. The constant term has been removed from the sum, so this case of equality is only possible if and only if $\mu = C\omega$ for some constant C.

□

We then let h be a general smooth metric. The following lemma follows immediately from the identity (45).

LEMMA **14.** *Let h be any smooth Hermitian metric, and let ω be a Kähler form with $\int_{\mathbb{CP}^1} \omega = \int_{\mathbb{CP}^1} \omega_h$. Then, their energy forms are related by*

$$\langle \mu, \nu \rangle_{\omega_h} = \langle \mu, \nu \rangle_\omega + \nu(\mathbb{CP}^1) \int U_\omega^{\omega_h} d\mu + \mu(\mathbb{CP}^1) \int U_\omega^{\omega_h} d\nu.$$

It follows that $-\mathcal{E}_h$ is strictly convex on $\mathcal{M}(\mathbb{CP}^1)$.

□

4.5. Green's function and L^2 norms.

LEMMA **15.** *Let G_h be the Green's function relative to ω_h. Then,*

$$(i)\ \ e^{\int_{\mathbb{CP}^1} G_h(z,w)dd^c \log ||s(w)||^2_{h^N}} = ||s||^2_{h^N}(z) e^{-\int_{\mathbb{CP}^1} \log ||s||^2_{h^N}(z)\omega_h}$$

and

$$(ii)\ \ \int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w)dd^c\log||s(z)||^2_{h^N} \boxtimes dd^c \log ||s(w)||^2_{h^N} \ = \ \int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w) Z_s \boxtimes Z_s.$$

Proof. The first point was proved in Lemma 9. Concerning the second point, we write $\frac{1}{N}dd^c \log ||s(z)||^2_{h^N} = Z_s - \omega_h$ (recall the Poincaré-Lelong formula (2.1)) . Then

$$\begin{aligned}
(ii) \ &= \ \int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w)(Z_s-\omega_h)\boxtimes(Z_s-\omega_h)\\
&= \ \int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w)Z_s\boxtimes Z_s\\
&\quad -2\int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w)Z_s\boxtimes\omega_h + \int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w)\omega_h\boxtimes\omega_h\\
&= \ \int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w)Z_s\boxtimes Z_s,
\end{aligned}$$

since $\int G_h(z,w)\omega_h = 0$ when integrating in either z or w. By Proposition 10, $G_h \in L^1(\mathbb{CP}^1,\omega_h)$; the integral over $\mathbb{CP}^1\times\mathbb{CP}^1\backslash D$ in the last terms is the same as over $\mathbb{CP}^1 \times \mathbb{CP}^1$. □

COROLLARY **16.** *We have:*

$$e^{\int_{\mathbb{CP}^1\times\mathbb{CP}^1\backslash D} G_h(z,w)dd^c\log||s_\zeta(z)||^2_{h^N}\boxtimes dd^c\log||s_\zeta(w)||^2_{h^N}} \ = \ e^{\sum_{i\neq j} G_h(\zeta_i,\zeta_j)}$$

4.6. Completion of proof of Proposition 3.

We now complete the proof of Proposition 3, which was started in §3.3. The purpose of this section is to convert the local expression (23) (see also (24)) for the joint probability current into a global invariant expression. We prove:

LEMMA **17.** *Let $h = e^{-\varphi}$ be a smooth Hermitian metric on $\mathcal{O}(1)$, and let ω_h, G_h be as above. Let $s_\zeta(z) = \prod_{j=1}^N (z-\zeta_j)e^N$. Then, the joint probability current is given by:*

$$\begin{aligned}
\frac{|\det \mathcal{A}_N(h,\nu)|^2|\Delta(\zeta_1,\ldots,\zeta_N)|^2\prod_{j=1}^N d^2\zeta_j}{\left(\int_{\mathbb{CP}^1}\prod_{j=1}^N|(z-\zeta_j)|^2e^{-N\varphi}d\nu(z)\right)^{N+1}} \ &= \ \frac{\exp\left(\frac{1}{2}\sum_{i\neq j}G_h(\zeta_i,\zeta_j)\right)}{\left(\int_{\mathbb{CP}^1} e^{\int_{\mathbb{CP}^1} G_h(z,w)d\mu_\zeta(w)}d\nu(z)\right)^{N+1}}\left(\prod_{j=1}^N e^{-2\varphi(\zeta_j)}d^2\zeta_j\right)\\
&\times \ |\det\mathcal{A}_N(h,\nu)|^2 e^{(-\frac{1}{2}N(N-1)+N(N+1))E(h)},
\end{aligned}$$

where $E(h)$ is defined in (7) and $\mathcal{A}_N$ is defined in (32). Moreover, $\prod_{j=1}^N e^{-2\varphi(\zeta_j)}d^2\zeta_j$ extends to a global smooth (N,N) form κ_N on $(\mathbb{CP}^1)^N$.

Proof. We first claim that

$$|\Delta(\zeta)|^2 = \exp\left(\sum_{i<j}G_h(\zeta_i,\zeta_j)\right)\exp\left((N-1)\sum_j\varphi(\zeta_j)-\frac{1}{2}(N-1)NE(h)\right). \tag{48}$$

Indeed, by Lemma 8,

$$2\log|z-w| = G_h(z,w)+\varphi(z)+\varphi(w)-E(h).$$

We note that $\log|\Delta(\zeta)|^2 = 2\sum_{i<j}\log|\zeta_i - \zeta_j|$ and that

$$\begin{array}{lll} 2\sum_{i<j}\log|\zeta_i-\zeta_j| & = & \sum_{i<j} G_h(\zeta_i,\zeta_j) + \sum_{i<j}(\varphi(\zeta_i)+\varphi(\zeta_j)) - E(h)) \\ & = & \sum_{i<j} G_h(\zeta_i,\zeta_j) + (N-1)\sum_j \varphi(\zeta_j) - \frac{1}{2}(N-1)NE(h) \\ & = & \sum_{i<j} G_h(\zeta_i,\zeta_j) + N(N-1)\int \varphi d\mu_\zeta - \frac{1}{2}(N-1)NE(h). \end{array}$$

We then convert the denominator into the Green's function expression by the identities

$$\begin{array}{lll} \int_{\mathbb{CP}^1}\prod_{j=1}^N |(z-\zeta_j)|^2 e^{-N\varphi} d\nu(z) & = & \int_{\mathbb{CP}^1} ||s_\zeta(z)||^2_{h^N} d\nu(z) \\ & = & \left(\int_{\mathbb{CP}^1} e^{\int_{\mathbb{CP}^1} G_h(z,w) dd^c \log||s_\zeta(w)||^2_{h^N}} d\nu\right) e^{\int_{\mathbb{CP}^1}\log||s_\zeta||^2_{h^N}(z)\omega_h} \\ & = & \left(\int_{\mathbb{CP}^1} e^{N\int_{\mathbb{CP}^1} G_h(z,w) d\mu_\zeta(w)} d\nu\right) e^{\int_{\mathbb{CP}^1}\log||s_\zeta||^2_{h^N}(z)\omega_h} \end{array} \tag{49}$$

by (22) and Lemma 15 (i). Further, by Lemma 9,

$$\int \log||s_\zeta(w)||^2_{h^N}\omega_h = N(\int \varphi d\mu_\zeta - E(h)).$$

We now raise the denominator (49) to the power $-(N+1)$ and multiply by (48) to obtain the Green's expression

$$\frac{\exp\left(\sum_{i<j} G_h(\zeta_i,\zeta_j)\right)}{\left(\int_{\mathbb{CP}^1} e^{\int_{\mathbb{CP}^1} G_h(z,w)d\mu_\zeta(w)} d\nu(z)\right)^{N+1}}$$

multiplied by the exponential of

$$(N-1)N\int\varphi d\mu_\zeta - \tfrac{1}{2}(N-1)NE(h) - N(N+1)(\int\varphi d\mu_\zeta - E(h)).$$

We note the cancellation in the N^2 term of $\int\varphi d\mu_\zeta$, leaving $-2N\int\varphi d\mu_\zeta = -2\sum_j\varphi(\zeta_j)$ This gives the stated result. The last statement follows from Lemma 5. ☐

4.7. **The approximate rate function I_N.** Lemma 17 expresses the joint probability $\vec{K}^N_n(\zeta_1,\dots,\zeta_N)$ as a geometric $(N+1,N+1)$ form on configuration space. In order to extract a rate function, we further express it as a functional of the measures μ_ζ. We introduce the following functionals.

Definition: Let $\zeta \in (\mathbb{CP}^1)^{(N)}$ and let μ_ζ be as in (1). Let $D = \{(z,z) : z \in \mathbb{CP}^1\}$ be the diagonal. Put:

$$\begin{cases} \mathcal{E}^h_N(\mu_\zeta) = \int_{\mathbb{CP}^1\times\mathbb{CP}^1\setminus D} G_h(z,w)d\mu_\zeta(z)d\mu_\zeta(w), \\ J^{h,\nu}_N(\mu_\zeta) = \log||e^{U^{\mu_\zeta}_h}||_{L^N(\nu)} \end{cases}$$

(Here, as before, with a slight abuse of notation we write $\|g\|_{L^N(\nu)} = \left(\int_{\mathbb{CP}^1}|g(\zeta)|^N d\nu(\zeta)\right)^{1/N}$.)

LEMMA **18.** *We have*

$$\vec{K}^N_n(\zeta_1,\dots,\zeta_N) = \frac{1}{\hat{Z}_N(h)} e^{-N^2\left(-\frac{1}{2}\mathcal{E}^h_N(\mu_\zeta)+\frac{N+1}{N}J^{h,\nu}_N(\mu_\zeta)\right)}\kappa_N$$

Proof. We are simply rewriting
$$\frac{\exp\left(\frac{1}{2}\sum_{i\neq j} G_h(\zeta_i,\zeta_j)\right)}{\left(\int_{\mathbb{CP}^1} e^{\int_{\mathbb{CP}^1} G_h(z,w)d\mu_\zeta(w)}d\nu(z)\right)^{N+1}} = e^{-N^2 I_N(\mu_\zeta)},$$
on the right side of Lemma 17 and leaving the other factors as they are. Then,
$$\begin{aligned} I_N(\mu_\zeta) &= -\tfrac{1}{N^2}\sum_{i\neq}\tfrac{1}{2}G_h(\zeta_i,\zeta_j) + \tfrac{N+1}{N^2}\log\left(\int_{\mathbb{CP}^1} e^{N\int_{\mathbb{CP}^1} G_h(z,w)d\mu_\zeta}d\nu(z)\right) \\ &= -\tfrac{1}{N^2}\tfrac{1}{2}\int_{\mathbb{CP}^1\times\mathbb{CP}^1\setminus D} G_h(z,w)d\mu_\zeta(z)d\mu_\zeta(w) + \tfrac{N+1}{N^2}\log\left(\int_{\mathbb{CP}^1} e^{NU_h^{\mu_\zeta}(z)}d\nu(z)\right) \\ &= -\tfrac{1}{N^2}\left(-\tfrac{1}{2}\mathcal{E}_N^h(\mu_\zeta) + \tfrac{N(N+1)}{N^2}J_N^{h,\nu}(\mu_\zeta)\right). \end{aligned}$$
□

5. Weighted equilibrium measures

In this section, we define the notion of weighted equilibrium measure $\nu_{h,K}$ of a non-polar compact set K with respect to a Hermitian metric h and prove that it is unique. In fact, there are two characterizations of $\nu_{h,K}$:

(i) $\nu_{h,K}$ is the minimizer of the Green's energy functional among measures supported on K.
(ii) The potential of $\nu_{h,K}$ is the maximal ω_h-subharmonic function of K.

We will need both characterizations in order to prove that the unique minimizer of the function $I^{h,K}$ of (14) is $\nu_{h,K}$. The problem is that $I^{h,K}$ differs significantly from the Green's energy on K and it is not obvious that they have the same minimizer.

In the classical case of weighted potential theory on $\mathbb{C}$, the equivalence of the two definitions is proved in [ST], especially in the appendix by T. Bloom. Their framework of admissible weights on $\mathbb{C}$ does not quite apply directly to the present setting of smooth Hermitian metrics on $\mathcal{O}(1)$ and potential theory on $\mathbb{CP}^1$. The second definition (ii) is assumed in work on potential theory on Kähler manifolds, e.g. as in [GZ], Definition 4.1. Only recently in [BB] have equilibrium measures been considered in terms of energy minimization. As a result, there is no simple reference for the facts we need, although their proofs are often small modifications of known proofs in the weighted case on $\mathbb{C}$. In that event, we only sketch the proof and refer the reader to the literature.

5.1. **Equilibrium measures as energy minimizers.** We now justify the first definition (i) by showing that there exists a unique energy minimizer (or maximizer, depending on the sign of the energy functional) among measures supported on a non-polar set K. We further prove that weighted equilibrium measures $\nu_{h,K}$ are unique and are supported on K (Proposition 19). We recall that K is a polar set if $\mathcal{E}_h(\mu) = -\infty$ for every finite non-zero Borel measure μ supported in K. In particular, a set satisfying (13) is non-polar.

Thus, we fix a compact non-polar subset $K \subset \mathbb{CP}^1$ and consider the restriction of the energy functional $\mathcal{E}_h : \mathcal{M}(K) \to \mathbb{R}$ to probability measures supported on K.

Proposition **19.** *If $K \subset \mathbb{CP}^1$ is non-polar, then $\mathcal{E}_h$ is bounded above on $\mathcal{M}(K)$. It has a unique maximizer $\nu_{K,h} \in \mathcal{M}(K)$.*

We denote its potential, the *weighted equilibrium potential*, by

$$U_h^{\nu_{K,h}}(z) = \int G_h(z,w)d\nu_{K,h}(w). \tag{50}$$

Proof. We begin by sketching the proof in the case where $\omega = \omega_h$ is a Kähler metric. In this case, the proof follows the standard lines of [Ran] (Theorem 3.3.2 and Theorem 3.7.6) or [ST], Theorem 1.3 and particularly Theorem 5.10. Existence follows from the upper semi-continuity of $\mathcal{E}_h$, which holds exactly as in the local weighted case.

Uniqueness by the method of [ST], Theorem I.1.3 or Theorem II.5.6 uses the non-positivity of the weighted logarithmic energy norm (Lemma I.1.8 of [ST]) or of the Green's energy norm (Theorem II.5.6). This argument applies directly to $\mathcal{E}_h$ when $\omega = \omega_h$ is Kähler : one assumes for purposes of contradiction that there exist two energy maximizers μ, ν of mass one. Then it follows by the argument of Theorem I. 1.3 (b) of [ST] that $||\mu - \nu||_\omega^2 = 0$; so $\mu - \nu = C\omega$. But integration over $\mathbb{CP}^1$ shows that $C = 0$, proving that $\mu = \nu$.

We then consider a general smooth Hermitian metric h. Since $\mathcal{E}_h$ is bounded above and $\mathcal{M}(K)$ is closed and hence compact, there exist measures in $\mathcal{M}(K)$ which maximize the energy $\mathcal{E}_h$. We now prove uniqueness:

LEMMA **20.** *If $K \subset \mathbb{CP}^1$ is non-polar, and h is any smooth metric, then $\mathcal{E}_h$ has a unique maximizer $\nu_{K,h} \in \mathcal{M}(K)$.*

Proof. We put

$$V_{\omega_h}(K) = \max\{\mathcal{E}_h(\mu) : \mu \in \mathcal{M}(K)\} < \infty.$$

To prove uniqueness, as before let ω be a Kähler metric in the same cohomology class as ω_h and observe that, for any signed measure $\mu - \nu$ given by a difference of two elements of $\mathcal{M}(\mathbb{CP}^1)$, hence satisfying $\int_{\mathbb{CP}^1} d(\mu - \nu) = 0$, we have

$$||\mu - \nu||_{\omega_h}^2 = ||\mu - \nu||_\omega^2.$$

Indeed, from (45),

$$\begin{aligned} ||\mu - \nu||_{\omega_h}^2 &= \textstyle\int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_h(z,w) d(\mu - \nu) \otimes d(\mu - \nu) \\ &= \textstyle\int_{\mathbb{CP}^1 \times \mathbb{CP}^1} (G_\omega(z,w) + U_h^\omega(z) + U_h^\omega(w) - \int U_h^\omega \omega)) d(\mu - \nu) \otimes d(\mu - \nu) \\ &= \textstyle\int_{\mathbb{CP}^1 \times \mathbb{CP}^1} G_\omega(z,w) d(\mu - \nu) \otimes d(\mu - \nu) = ||\mu - \nu||_\omega^2. \end{aligned}$$

Hence, the energy form is negative semi-definite on the subspace of signed measures $\mu - \nu$ where μ, ν are positive and of the same mass.

Suppose that $\mu, \nu \in \mathcal{M}(K)$ and that both are maximizers of $\mathcal{E}_h$ on $\mathcal{M}(K)$. Then $\int_{\mathbb{CP}^1} d(\mu - \nu) = 0$ and hence $||\mu - \nu||_\omega^2 \leq 0$. Equality holds if and only if $\mu - \nu = C\omega$ for some C, and the fact that $\int d(\mu - \nu) = 0$ implies $C = 0$. But

$$||\frac{1}{2}(\mu + \nu))||_{\omega_h}^2 + ||\frac{1}{2}(\mu - \nu)||_{\omega_h}^2 = \frac{1}{2}(\mathcal{E}_h(\mu) + \mathcal{E}_h(\nu)) = V_{\omega_h}(K).$$

Since $\mathcal{E}_h(\sigma) \leq V_{\omega_h}(K)$ for any $\sigma \in \mathcal{M}(\mathbb{CP}^1)$, it follows that $||\mu - \nu||_\omega^2 = 0$ and hence $\mu = \nu$. This completes the proof of uniqueness.

$\square$

This completes the proof of Proposition 19. $\square$

Definition: The weighted capacity of K with respect to h (or equivalently ω_h) is defined by

$$\mathrm{Cap}_h(K) = e^{\sup\{\mathcal{E}_h(\mu):\mu\in\mathcal{M}(K)\}} = e^{\mathcal{E}_h(\nu_{K,h})}. \tag{51}$$

5.2. **Equilibrium measure and subharmonic envelopes.** We now discuss the second characterization (ii) of equilibrium measures (see the beginning of §5) and prove that it is equivalent to the first.

Given a closed real $(1,1)$ form ω (not necessarily a Kähler form), and a compact subset $K \subset \mathbb{CP}^1$, define the global extremal function $V^*_{K,\omega}$ as the upper semi-continuous regularization of

$$V_{K,\omega}(z) := \sup\{u(z) : u \in SH(\mathbb{CP}^1, \omega, K)\},$$

where

$$SH(\mathbb{CP}^1, \omega, K) := \{u \in SH(\mathbb{CP}^1, \omega) : u \le 0 \text{ on } K\}.$$

(See (39) for the definition of $SH(\mathbb{CP}^1, \omega)$.)

In what follows we take $\omega = \omega_h$ and replace the subscript ω_h by h. The important properties of $V^*_{K,h}$ and $\nu_{K,h}$ are the following, a special case of Theorem 4.2 of [GZ]:

THEOREM **21.** *Let $K \subset \mathbb{CP}^1$ be a Borel set. If K is non-polar, then $V^*_{K,h} \in SH(\mathbb{CP}^1, \omega_h)$ and satisfies:*

(1) $\nu_{K,h} = 0$ *on* $\mathbb{CP}^1\backslash\overline{K}$.
(2) $V^*_{K,h} = 0$ *quasi-everywhere on* $\mathrm{Supp}\,\nu_{K,h}$ *and in the interior of* K*;*
(3) $\int_{\overline{K}} \nu_{K,h} = \int \omega_h$ $(= 1)$.

The following Proposition relates $V^*_{K,h}$ to the potential (50) of the equilibrium measure of Proposition 19.

PROPOSITION **22.** *Let $K \subset \mathbb{CP}^1$ be a non-polar compact subset and let ω_h be a smooth $(1,1)$ form with $\int_{\mathbb{CP}^1} \omega_h = 1$. Then,*

$$\nu_{K,h} = dd^c V^*_{K,h} + \omega_h.$$

Moreover,

$$U_h^{\nu_{K,h}} = V^*_{K,h} - \int_{\mathbb{CP}^1} V^*_{K,h}\omega_h. \tag{52}$$

In particular, with $F_{K,h} = \int_{\mathbb{CP}^1} V^*_{K,h}\omega_h$, we have that $U_h^{\nu_{K,h}} = -F_{K,h}$ quasi-everywhere on the support of $\nu_{K,h}$.

Proof. We only sketch the proof, which is barely different from the case of admissible potential theory on $\mathbb{C}$ [ST].

The second statement (52) implies the first. It shows that both $U_h^{\nu_{K,h}}$ and $V^*_{K,h}$ belong to $SH(\mathbb{CP}^1, \omega_h)$ and are potentials for $\nu_{K,h}$, i.e.

$$dd^c U_h^{\nu_{K,h}} = \nu_{K,h} - \omega_h = dd^c V^*_{K,h}.$$

The potentials must differ by a constant, which is determined by integrating with respect to ω_h and using that $\int U_h^{\mu}\omega_h = 0$ for any μ. The proof is essentially the same as in the classical unweighted case (see Lemma 2.4 of Appendix B.2 of [ST]).

It therefore suffices to prove (52). The proof in the case of weighted potential theory on $\mathbb{C}$ is given in Theorem I.4.1 of [ST], as sharpened in Appendix B, Lemma 2.4 of [ST]. The main ingredients are the so-called principle of domination (see [ST], I.3), and the Frostman type theorem that $U_h^{\nu_{K,h}} \geq F_{K,h}$ q.e. on K and $U_h^{\nu_{K,h}} \leq F_{K,h}$ on $\mathrm{Supp}\,\nu_{K,h}$, hence $U_h^{\nu_{K,h}} = F_{K,h}$ q.e. on $\mathrm{Supp}\,\nu_{K,h}$ (see [ST], Theorem I.1.3 (d)-(f)). □

5.3. **Thin points and capacity.** Our assumption on K =suppν is that it is non-h-thin at all of its points. We now define thin-ness with respect to h and its Green's function G_h.

A set E is said to be h-*thin* at $x_0 \in \mathbb{CP}^1$ iff either of the following occur:

- x_0 is not a limit point of E; or,
- There exist $\epsilon > 0$, $\eta > 0$, and a potential U_h^μ so that $U_h^\mu(x_0) > -\infty$ and $U_h^\mu(x) \leq U_h^\mu(x_0) - \eta$ for all $x \in E \cap D_\epsilon(x_0)\backslash\{x_0\}$.

See [Lan] (5.3.2), page 307 (but recall that all definitions in [Lan] differ by a multiplicative $-$ sign from the definitions used here). Thin-ness is a local notion. Although we define it in terms of ω_h-potentials, a point $x \in \mathbb{C}$ is a non-h-thin point of E if and only it is non-thin in the sense of logarithmic potential theory. The definition above also applies to the h-thin-ness of E at ∞. Thin-ness can be characterized in terms of the fine topology, i.e. the weakest topology on $\mathbb{CP}^1$ with respect to which all ω_h-subharmonic functions are continuous. Namely, E is non-thin at x if and only if x is a fine limit point of E. We refer to [Lan], Definition Ch. V §3 (5.3.1) or [Ran, D] for background on thin-ness of subsets $E \subset \mathbb{C}$.

We will need the following properties of thin sets (they hold for h-thinness precisely in the same way as for thin-ness):

- A subset of a thin set at x_0 is also thin at x_0; the union of two thin sets at x_0 is thin at x_0;
- A set of h-capacity zero is the same as a polar set.
- A set which is thin at all of its points is polar. A polar F_σ set is thin at all of its points (see Ransford Theorem 3.8.2 and Cor 3.8.7.)

We further recall:

LEMMA **23.** *(see* [ST]*, Corollary 6.11 of Ch. I, or the Corollary to Theorem 3.7 of* [Lan]*, Ch. III §2) If $S \subset \mathbb{C}$ is compact and of positive capacity, then there exists a positive, finite measure ν, with support included in S, so that $U^\nu \in C(\mathbb{C})$.*

The Lemma and proof extend with no essential change to Cap_h and U_h^ν on $\mathbb{CP}^1$.

6. Rate function and equilibrium measure

We continue to fix a pair (h, ν) where h is a smooth Hermitian metric on $\mathcal{O}(1)$ and where ν is a measure satisfying (11). The purpose of this section is to prove that the rate function (15) of the LDP of Theorem 1 is a *good rate function* and also to prove that its unique minimizer is the equilibrium measure for (h, K). That is, we prove:

PROPOSITION **24.** *The function $I^{h,K}$ of (14) has the following properties:*

1. *It is a lower-semicontinuous functional.*
2. *It is strictly convex.*
3. *Its unique minimizer is the equilibrium measure $\nu_{h,K}$.*
4. *Its minimum value equals $\frac{1}{2}\log \mathrm{Cap}_h(K)$.*

We prove (1) and (2) now; (3) and (4) will be proved in Section 6.1. We begin with the following elementary consequence of Proposition 10.

LEMMA **25.** *For each $z \in \mathbb{CP}^1$, the function $\mu \to U_h^\mu(z)$ is an upper semi-continuous function from $\mathcal{M}(\mathbb{CP}^1)$ to $\mathbb{R} \cup \{-\infty\}$. Further, so is the function $\mu \to \mathcal{E}_h(\mu)$.*

Proof. Fix $M \in \mathbb{R}$ and define $G_h^M(z,w) = G_h(z,w) \vee (-M)$. By Proposition 10, G_h^M is continuous on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Set

$$U_h^{\mu,M}(z) = \int_{\mathbb{CP}^1} G_h^M(z,w)d\mu(w)\,, \quad \mathcal{E}_h^M(\mu) = \int_{\mathbb{CP}^1\times\mathbb{CP}^1} G_h^M(z,w)d\mu(z)d\mu(w)\,. \tag{53}$$

For fixed z, it follows that $\mu \to U_h^{\mu,M}(z)$ and $\mu \to \mathcal{E}_h^M(\mu)$ are continuous on $\mathcal{M}(\mathbb{CP}^1)$. Since $U_h^\mu(z) = \inf_M U_h^{\mu,M}(z)$ and $\mathcal{E}_h(\mu) = \inf_M \mathcal{E}_h^M(\mu)$, the claimed upper semi-continuity follows. □

We next have the following.

LEMMA **26.** (1) *The function $J^{h,K}(\mu) = \sup_{z\in K} U_h^\mu(z)$ is upper semi-continuous.*
(2) *Assume that all points of K are regular. Then $J^{h,K}(\mu)$ is also lower semi-continuous.*

Proof. **(i) Upper semi-continuity**

We begin by proving the upper semi-continuity. Let $\mu_n \to \mu^*$ weakly in $\mathcal{M}(\mathbb{CP}^1)$. Fix $M \in \mathbb{R}$ and recall that $G_h^M(\cdot,\cdot)$ is continuous on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Therefore, the map $(z,\mu) \to U_h^{\mu,M}(z)$ is continuous. Because K is compact, $\mu \mapsto \sup_{z\in K} U_h^{\mu,M}(z)$ is therefore continuous. Thus,

$$J^{h,K}(\mu_n) = \sup_{z\in K} U_h^{\mu_n}(z) \le \sup_{z\in K} U_h^{\mu_n,M}(z) \to_{n\to\infty} \sup_{z\in K} U_h^{\mu^*,M}(z)\,.$$

Since $U_h^{\mu^*,M}(z) \to_{M\to\infty} U_h^{\mu^*}(z)$ for any z by monotone convergence, we have

$$\sup_{z\in K} U_h^{\mu^*,M}(z) \to_{M\to\infty} \sup_{z\in K} U_h^{\mu^*}(z) = J^{h,K}(\mu)\,.$$

Combining the last two displays completes the proof of (1).

(ii) Lower semi-continuity

Let K be a set all of whose points are non-thin. Suppose that $\mu_n \to \mu$. We claim that

$$\liminf_{n\to\infty} \sup_K U_h^{\mu_n} \ge \sup_{z\in K} U_h^\mu(z) := a.$$

(Recall that $a < \infty$.) For any $\epsilon > 0$ introduce the set

$$A_\epsilon = \{z \in \mathbb{CP}^1 : U_h^\mu \ge a - \epsilon\}.$$

A_ϵ is a closed set since U_h^μ is upper semi-continuous. We claim the following

$$\forall \epsilon > 0, \ \ \mathrm{Cap}_h(A_\epsilon \cap K) > 0. \tag{54}$$

To prove the claim, let z^* be a point where U_h^μ attains its maximum on K (such a point exists by the upper semicontinuity of $z \mapsto U_h^\mu(z)$ and the compactness of K). By assumption, z^* is a non-h-thin point. Since

$$U_h^\mu < a - \epsilon, \ \ \text{on } K\backslash A_\epsilon,$$

$K\backslash A_\epsilon$ is h-thin at z^*. Suppose that there exists $\epsilon_0 > 0$ so that $\mathrm{Cap}_h(A_{\epsilon_0} \cap K) = 0$. A closed set of capacity zero is h- thin at each of its points (as mentioned in §5.3, the proof in

[Ch], Corollary on page 92 or [Ran], Theorem 3.8.2 applies to h-thin sets with no essential change). Since

$$K = K \backslash A_{\epsilon_0} \cup (K \cap A_{\epsilon_0}),$$

and since the union of two sets h-thin at z^* is h-thin at z^*, we see that K is h-thin at z^*. This contradicts the assumption that K is non-h-thin at z^*, and thus proves (54).

We now complete the proof of the lemma. Let $\mathbf{1}_{A_\epsilon \cap K}$ denote the characteristic function of $A_\epsilon \cap K$. Since U_h^μ is upper semi-continuous, $A_\epsilon \cap K$ is compact. By (54), it has positive capacity. It follows by Lemma 23 there exists a positive measure $\nu_{\epsilon,\mu,K}$ supported on $A_\epsilon \cap K$ whose potential $U_h^{\nu_{\epsilon,\mu,K}}$ is continuous.

We have,

$$\begin{aligned}
\lim_{n\to\infty} \int_{A_\epsilon} U_h^{\mu_n}(z) d\nu_{\epsilon,\mu,K}(z) &= \lim_{n\to\infty} \int U_h^{\nu_{\epsilon,\mu,K}}(z) d\mu_n(z) \\
&= \int U_h^{\nu_{\epsilon,\mu,K}} d\mu(z) \\
&= \int_{A_\epsilon} U_h^{\mu}(z) d\nu_{\epsilon,\mu,K}(z).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nu_{\epsilon,\mu,K}(A_\epsilon \cap K) \liminf_{n\to\infty} \sup_K U_h^{\mu_n} &\geq \liminf_{n\to\infty} \int_{A_\epsilon} U_h^{\mu_n}(z) d\nu_{\epsilon,\mu,K}(z) \\
&= \int_{A_\epsilon} U_h^{\mu}(z) d\nu_{\epsilon,\mu,K}(z) \geq (a-\epsilon)\nu_{\epsilon,\mu,K}(A_\epsilon \cap K).
\end{aligned}$$

Since $\nu_{\epsilon,\mu,K}(A_\epsilon \cap K) > 0$ and since ϵ is arbitrary, this finishes the proof. □

Remark: We note that if $d\nu$ is any measure on K whose potential U^ν is continuous, then $U_h^{\mathbf{1}_{A_\epsilon \cap K}\nu}$ is automatically continuous. Indeed, $U_h^{\mathbf{1}_{A_\epsilon \cap K}\nu}$ is upper semi-continuous, so we only need to prove that it is lower semi-continuous. But

$$U_h^{\mathbf{1}_{A_\epsilon \cap K}\nu} = U_h^\nu - U_h^{\nu - \mathbf{1}_{A_\epsilon \cap K}\nu},$$

and the first term on the right is continuous and the second, being the opposite of a potential, is lower semi-continuous.

A consequence of Lemma 26 is that $J^{h,K}(\cdot)$ is bounded on $\mathcal{M}(\mathbb{CP}^1)$, and thus $I^{h,K}(\cdot)$ is well-defined. Further, we have the following.

Lemma **27.** *The function $\tilde{I}^{h,K}(\cdot)$ on $\mathcal{M}(\mathbb{CP}^1)$ is a rate function.*

Proof. By Lemma 25 and Lemma 26, the function

$$-\frac{1}{2}\mathcal{E}_h(\cdot) + J^{h,K}(\cdot)$$

is well defined on $\mathcal{M}(\mathbb{CP}^1)$, bounded below, and lower semi-continuous. This implies the claim. □

Next we prove strict convexity of the rate function. Strict convexity of the (unweighted) logarithmic energy is well-known [ST, BG, BB].

Lemma **28.** *$I^{h,K}$ is a strictly convex function on $\mathcal{M}$ (with possible values $+\infty$)*

Proof. Strict convexity of the energy functional is proved in Proposition 12. To complete the proof, we note that the 'potential term' $J^{h,K}(\mu) = \sup_K U_h^\mu(z)$ is a maximum of affine functions of μ, hence is convex.

□

6.1. The global minimizer of $I^{h,K}$.

Since $I^{h,K}$ is lower semi-continuous it has a minimum on $\mathcal{M}(\mathbb{CP}^1)$ and also on each closed ball $B(\sigma,\delta) \subset \mathcal{M}(\mathbb{CP}^1)$. In this section we show that the global minimum is the equilibrium measure $\nu_{K,h}$ for the data (h,ν) defining $I^{h,K}$.

The equilibrium measure $\nu_{K,h}$ is the unique maximizer of $\mathcal{E}_h(\mu)$ on $\mathcal{M}(K)$. Our function differs from this constrained function in not being constrained to $\mathcal{M}(K)$ but rather possessing the term $\sup_K U_h^\mu$. We need to show that this term behaves like a 'Lagrange multiplier' enforcing the constraint. Unfortunately, it is not 'smooth' as a function of μ, so we cannot use calculus alone to demonstrate this.

LEMMA **29.** *The global minimizer of $I^{h,K}$ is $\nu_{K,h}$. The global minimum is $\frac{1}{2}\log \mathrm{Cap}_h(K)$.*

Proof. We write

$$\begin{aligned} 2I^{h,K}(\mu) &= -\mathcal{E}_h(\mu) + 2\sup_K U_h^\mu \\ &= -\int_{\mathbb{CP}^1} (U_h^\mu - \sup_K U_h^\mu)(dd^c U_h^\mu + \omega_h) + \sup_K U_h^\mu. \end{aligned}$$

We claim that $\int_{\mathbb{CP}^1}(U_h^\mu - \sup_K U_h^\mu)(dd^cU_h^\mu + \omega_h) - \sup_K U_h^\mu$ is maximized when $\mu = \nu_{K,h}$ so that $U_h^\mu - \sup_K U_h^\mu = V_{K,h}^*$. By definition, $U_h^\mu - \sup_{z\in K} U_h^\mu \le 0$ on K. Hence, $U_h^\mu - \sup_{z\in K} U_h^\mu \le V_{K,h}^*$. Since $dd^cU_h^\mu + \omega_h$ is the positive measure $d\mu$,

$$\begin{aligned} -2I^{h,K}(\mu) &= \int_{\mathbb{C}} (U_h^\mu - \sup_K U_h^\mu)(dd^c U_h^\mu + \omega_h) - \sup_K U_h^\mu \\ &\le \int V_{K,h}^*(dd^cU_h^\mu + \omega_h) - \sup_K U_h^\mu \\ &= \int (U_h^\mu - \sup_{z\in K} U_h^\mu) dd^c V_{K,h}^* - \sup_K U_h^\mu + \int_{\mathbb{CP}^1} V_{K,h}^*\omega_h \\ &= \int (U_h^\mu - \sup_{z\in K} U_h^\mu)(dd^c V_{K,h}^* + \omega_h) + \int_{\mathbb{CP}^1} V_{K,h}^*\omega_h \\ &\le \int V_{K,h}^*(dd^c V_{K,h}^* + \omega_h) + \int_{\mathbb{CP}^1} V_{K,h}^*\omega_h \\ &= F_{K,\omega_h}. \end{aligned}$$

In the third line, we integrated dd^c by parts, and used that constants integrate to 0 against $dd^cV_{K,h}^*$. In the next to last line, we again use that $U_h^\mu - \sup_{z\in K} U_h^\mu \le V_{K,h}^*$. In the last equality, we used Proposition 22.

Since

$$\int_{\mathbb{CP}^1} (U_h^{\nu_{h,K}} - \sup_K U_h^{\nu_{h,K}})(dd^c U_h^{\nu_{h,K}} + \omega_h) - \sup_K U_h^{\nu_{h,K}} = \int V_{K,h}^*(dd^c V_{K,h}^* + \omega_h) + F_{K,\omega_h},$$

we see that $I^{h,K}$ is is minimized by $\nu_{h,K}$.

One easily checks that all of the inequalities are equalities for $\nu_{K,h}$. When K is regular, $V^*_{K,h} = (U_h^{\nu_{K,h}} - \sup_K U_h^{\nu_{K,h}})$ is continuous. We can determine the sup by integrating both sides against ω_h as in Proposition 22:

$$\int V^*_{K,h}\omega_h = -\sup_K U_h^{\nu_{K,h}}.$$

We have,

$$\begin{aligned} -2I^{h,K}(\nu_{K,h}) &= \int_{\mathbb{CP}^1} (U_h^{\nu_{K,h}} - \sup_K U_h^{\nu_{K,h}})(dd^c U_h^{\nu_{K,h}} + \omega_h) - \sup_K U_h^{\nu_{K,h}} \\ &= \int V^*_{K,h}(dd^c V^*_{K,h} + \omega_h) + \int_{\mathbb{CP}^1} V^*_{K,h}\omega_h = F_{K,\omega_h}. \end{aligned}$$

On the other hand, since $U_{\omega_h}^{\nu_{K,h}} = -F_{K,\omega_h}$ on K, we have that

$$\log \mathrm{Cap}_h(K) = \mathcal{E}_{\omega_h}(\nu_{K,h}) = \int U_{\omega_h}^{\nu_{K,h}} d\nu_{K,h} = -F_{K,\omega}.$$

This completes the proof. □

7. Large deviations theorems in genus zero: Proof of Theorem 1

In this section, we prove Theorem 1. We already know that $\tilde{I}^{h,K}$ is a good rate function. We still need to prove that it actually is the rate function of the large deviations principle. As in [BG] (Section 3), it is equivalent to prove that

$$-I(\sigma) := \limsup_{\delta\to 0} \limsup_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma,\delta)) = \liminf_{\delta\to 0} \liminf_{N\to\infty} \frac{1}{N^2} \log \mathbf{Prob}_N(B(\sigma,\delta)). \tag{55}$$

See Theorem 4.1.11 of [DZ].

The proof follows the approach in [BG, BZ] of large deviations principles for empirical measures of eigenvalues of certain random matrices. However, we take full advantage of the compactness of $\mathcal{M}(\mathbb{CP}^1)$, and of the properties of the Green function G_h, see Lemma 25 and Proposition 26.

We first prove the result without taking the normalizing constants $Z_N(h)$ of Proposition 3 into account. Then in §7.4, we determine the logarithmic asymptotics of the normalizing constants.

7.1. Heuristic derivation of $I^{h,K}$.

Since the proof of the large deviations principle is technical, we first give a formal or heuristic derivation of the rate function (17) from the expression for the joint probability distribution of zeros in Lemma 18 in the spirit of the discussion in §1.5. We then fill in the technical gaps to give a rigorous proof.

7.1.1. *Heuristic derivation.* Lemma 18 expresses $\vec{K}_n^N(\zeta_1,\ldots,\zeta_N)$ as a product of three factors: the normalizing constant $\frac{1}{\hat{Z}_N(h)}$, the factor $e^{-N^2\left(-\frac{1}{2}\mathcal{E}_N^h(\mu_\zeta)+\frac{N+1}{N}J_N^{h,\nu}(\mu_\zeta)\right)}$ and the integration measure $\prod_{j=1}^N e^{-2N\varphi(\zeta_j)}d^2\zeta_j$. The normalizing constant will be worked out asymptotically in §7.4 using the fact that $\vec{K}_n^N(\zeta_1,\ldots,\zeta_N)$ is a probability measure. The integration measure

$\prod_{j=1}^N e^{-2N\varphi(\zeta_j)} d^2\zeta_j$ is an invariantly defined smooth (N, N) on $(\mathbb{CP}^1)^N$ of finite mass independent of N, and thus does not contribute to the logarithmic asymptotics. Hence, only the factor $\left(-\frac{1}{2}\mathcal{E}_N^h(\mu_\zeta) + \frac{N+1}{N} J_N^{h,\nu}(\mu_\zeta)\right)$ contributes to the rate function.

The term $\mathcal{E}_N^h(\mu_\zeta)$ closely resembles the energy except that the diagonal D has been punctured out of the domain of integration, as it must since μ_ζ has infinite energy. It must be shown that the true energy is the correct limiting form when measuring log probabilities of balls of measures.

The second term satisfies,

$$\lim_{N\to\infty} J_N^{h,\nu}(\mu_\zeta) = \log ||e^{U_h^\mu}||_{L^N(\nu)} \uparrow \log ||e^{U_h^\mu}||_{L^\infty(\nu)} = \sup_K U_h^\mu$$

monotonically as $N \to \infty$. Thus, it is natural to conjecture that the rate function for large deviations of empirical measures is given by (17).

We now turn to the rigorous proof.

7.2. **Proof of the upper bound.** In this section, we prove the upper bound part of the large deviation principle, that is we prove that

$$\lim_{\delta\downarrow 0} \limsup_N \frac{1}{N^2} \log \, \mathbf{Prob}_N(B(\sigma, \delta)) \le -\tilde{I}^{h,K}(\sigma). \tag{56}$$

The first step is:

Lemma **30.** *Fix $\epsilon > 0$. If ν satisfies the Bernstein-Markov condition (11), then there exists a $N_0 = N_0(\epsilon)$ such that for all $N > N_0$ and all $\mu_\zeta \in \mathcal{M}(\mathbb{CP}^1)$,*

$$\log ||e^{U_h^{\mu_\zeta}}||_{L^N(\nu)} \ge \sup_{z\in K} U_h^{\mu_\zeta} - \epsilon \, .$$

Proof. We are assuming that, for all $s \in H^0(\mathbb{C}, L^N)$,

$$\sup_{z\in K} |s(z)|_{h^N} \le C_\epsilon e^{\epsilon N} \left(\int_K |s(z)|_{h^N}^2 d\nu(z) \right)^{1/2} . \tag{57}$$

By Lemma 9 we may write

$$|s_\zeta(z)|_{h^N}^2 = e^{NU_h^{\mu_\zeta}(z)} e^{N(\int \varphi d\mu_\zeta - E(h))}$$

Hence,

$$\begin{aligned} ||e^{U_h^{\mu_\zeta}}||_{L^N} e^{(\int \varphi d\mu_\zeta - E(h))} &= \left(\int_K |s_\zeta(z)|_{h^N}^2 d\nu(z)\right)^{1/N} \\ &\ge \left(C_\epsilon^{-1} e^{-N\epsilon} \sup_{z\in K} |s_\zeta(z)|_{h^N}^2\right)^{\frac{1}{N}} \\ &\implies \log ||e^{U_h^{\mu_\zeta}}||_{L^N} \ge \sup_{z\in K} U_h^{\mu_\zeta} - \epsilon + \tfrac{1}{N} \log C_\epsilon, \end{aligned}$$

for all $\epsilon > 0$. □

Write

$$\Theta_N = -\frac{1}{N^2} \log \hat{Z}_N(h) \, .$$

As we will see in the course of the proof of Lemma 4, $\Theta_N \to_{N\to\infty} \log \mathrm{Cap}_h(K)$.

By Lemma 17 and Lemma 18,

$$\frac{1}{N^2}\log\ \mathbf{Prob}_N(B(\sigma,\delta)) = \frac{1}{N^2}\log\int_{\zeta\in(\mathbb{CP}^1)^N:\mu_\zeta\in B(\sigma,\delta)} e^{-N^2 I_N(\mu_\zeta)}\kappa_N + \Theta_N. \tag{58}$$

Fix $M\in\mathbb{R}$ and let $G_h^M = G_h \vee (-M)$ be the truncated Green function. By Lemma 25, G_h^M is continuous on $\mathbb{CP}^1\times\mathbb{CP}^1$. Further, with notation as in (53),

$$\begin{aligned} -\frac{1}{N^2}\sum_{i<j} G_h(\zeta_i,\zeta_j) &\geq -\frac{1}{N^2}\sum_{i<j} G_h^M(\zeta_i,\zeta_j) \\ &\geq -\frac{1}{2}\int\int_{\mathbb{CP}^1\times\mathbb{CP}^1} G_h^M(z,w)d\mu_\xi(z)d\mu_\xi(w) - \frac{C(M)}{N} = \mathcal{E}_h^M(\mu_\xi) - \frac{C(M)}{N}, \end{aligned}$$

where the constant $C(M)$ does not depend on ξ. Using Lemma 30 (and also Corollary 11), we then have that for any $\epsilon>0$ and all $N>N_0(\epsilon)$,

$$\begin{aligned} \tfrac{1}{N^2}\log\ \mathbf{Prob}_N(B(\sigma,\delta)) &\leq \tfrac{1}{N^2}\log\textstyle\int_{\xi\in(\mathbb{CP}^1)^N:\mu_\xi\in B(\sigma,\delta)} e^{\frac{N^2}{2}\mathcal{E}_h^M(\mu_\xi)-N^2 J_h^K(\mu_\xi)}\kappa_N \\ &+ (\Theta_N + \tfrac{C'(M)}{N}+\epsilon), \end{aligned}$$

for some constant $C'(M)$ (recall Lemma 17 for the definition of the (N,N) form κ_N). It follows that

$$\limsup_N \frac{1}{N^2}\log\ \mathbf{Prob}_N(B(\sigma,\delta)) \leq \limsup_{N\to\infty}\Theta_N + \limsup_{\delta\downarrow 0}\sup_{\mu\in B(\sigma,\delta)} -\left(-\frac{1}{2}\mathcal{E}_h^M(\sigma)+J_h^K(\sigma)\right).$$

Here, we use that

$$\frac{1}{N^2}\log\int_{(\mathbb{CP}^1)^N}\kappa_N = O(\frac{\log N}{N}), \tag{59}$$

which follows from Lemma 5.

It then follows from the continuity of $\mathcal{E}_h^M(\sigma)$ and the lower semi-continuity of $J_h^K(\sigma)$ (see Lemma 26) that

$$\lim_{\delta\downarrow 0}\limsup_N \frac{1}{N^2}\log\ \mathbf{Prob}_N(B(\sigma,\delta)) \leq \lim_{N\to\infty}\Theta_N + \frac{1}{2}\mathcal{E}_h^M(\sigma) - J_h^K(\sigma)+\epsilon\,.$$

Since $\mathcal{E}_h^M(\sigma)\to\mathcal{E}_h(\sigma)$ as $M\to\infty$ by monotone convergence, and since ϵ is arbitrary, we obtain (56), and hence the upper bound in (55). □

7.3. **Proof of the lower bound.** In this section, we prove that

$$\lim_{\delta\downarrow 0}\liminf_N \frac{1}{N^2}\log\ \mathbf{Prob}_N(B(\sigma,\delta)) \geq -\tilde{I}^{h,K}(\sigma). \tag{60}$$

Together with the upper bound (56), this will show that in fact equality holds in (60), and will complete the proof of (55).

The strategy is again similar to [BG] and [BZ]. Note first that to prove (60), it is enough to find, for any σ with $I(\sigma)<\infty$, a sequence $\sigma_\epsilon \to_{\epsilon\to 0} \sigma$ weakly in $\mathcal{M}(\mathbb{CP}^1)$ such that $\tilde{I}^{h,K}(\sigma_\epsilon)\to\tilde{I}^{h,K}(\sigma)$ and (60) holds for σ_ϵ.

So, define $\sigma_\epsilon = e^{\epsilon\Delta_\omega}\sigma$ as in Lemma 33 of the appendix. By that lemma, the property $\tilde{I}^{h,K}(\sigma_\epsilon)\to_{\epsilon\to 0}\tilde{I}^{h,K}(\sigma)$ holds. It thus only remains to prove (60) when σ is replaced by σ_ϵ. Thus, the large deviations lower bound is a consequence of the following lemma.

Lemma **31.** *Let $\sigma = f\omega \in \mathcal{M}(\mathbb{CP}^1)$ with f a strictly positive and continuous function on $\mathbb{CP}^1$. Then,* (60) *holds.*

Proof. We follow [BZ], Lemma 2.5. It will be convenient to consider three charts on $\mathbb{CP}^1$, denoted W_0, W_1, W_2, with distance d_{W_i} on W_i, that cover $\mathbb{CP}^1$, and a constant R, in such a way that for any two points $z, w \in \mathbb{CP}^1$ there exists a chart $W_{z,w}$ with distance $d_{W_{z,w}}$ so that in local coordinates, $d(z,w) := d_{W_{z,w}}(z,w) \leq R$ (If more than one such chart exists for a given pair z, w, fix one arbitrarily as $W_{z,w}$. The charts W_0 can be taken as the standard chart $\mathbb{C}$, with W_1 and W_2 its translation to two fixed distinct points in $\mathbb{CP}^1$.)

Construct a sequence of discrete probability measures

$$d\sigma_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{Z_j} \in B(\sigma, \delta)$$

with the following properties:

(1) $\sigma_N \in B(\sigma, \delta/2)$ for all N large;
(2) $d(Z_i, Z_j) \geq \frac{C(\sigma,\delta)}{\sqrt{N}}$.

(Since in a local chart, σ possesses a bounded density with respect to Lebesgue's measure, such a sequence can be constructing by adapting to the local charts W_i the construction in the proof of Lemma 2.5 in [BZ].) Define

$$D_I^\eta = \{\zeta \in (\mathbb{CP}^1)^N : d(\zeta_j, Z_j) \leq \frac{\eta}{N}, \;\; j = 1, \ldots, N\}.$$

Then, for η small enough and all N large, all $\zeta \in D_I^\eta$ satisfy that $\mu_\zeta \in B(\sigma, \delta)$. Since $D_I^\eta \subset B(\sigma, \delta)$,

$$\mathbf{Prob}_N(B(\sigma,\delta)) \geq \int_{D_I^\eta} e^{-N^2 I_N(\mu_\zeta)} \kappa_N + \Theta_N. \tag{61}$$

(Recall Lemma 17 for the definition of the (N, N) form κ_N.) By Proposition 10 and our construction, there exists a constant $C_1 = C_1(\eta, \sigma)$ with $C_1 \to_{\eta\to 0} 0$ such that for any $\xi \in D_I^\eta$ and $i, j \in \{1, \ldots, N\}$, $i \neq j$,

$$G_h(\xi_i, \xi_j) \geq G_h(Z_i, Z_j) - C_1(\eta)\,.$$

For $\epsilon > 0$, set

$$\mathcal{E}_h^\epsilon(\sigma) := \int_{\mathbb{CP}^1\times\mathbb{CP}^1\setminus D_\epsilon} G_h(z,w) d\sigma(z) d\sigma(w)\,,$$

where $D_\epsilon = \{(z,w) \in (\mathbb{CP}^1 \times \mathbb{CP}^1) : d(z,w) < \epsilon\}$. We have by monotone convergence that

$$\mathcal{E}_h(\sigma) = \lim_{\epsilon\to 0} \mathcal{E}_h^\epsilon(\sigma).$$

Because G_h is continuous on $\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D_\epsilon$, we have that

$$N^{-2} \sum_{i\neq j, d(Z_i,Z_j)\geq\epsilon} G_h(Z_i, Z_j) \geq \mathcal{E}_h(\sigma) - C_2(\epsilon, \delta)\,,$$

where for fixed ϵ, $C_2(\epsilon,\delta) \to_{\delta\to 0} 0$. On the other hand, let $J_i(r) = \#\{j \in \{1, \ldots, N\} : j \neq i, d(Z_i, Z_j) \in [r/\sqrt{N}, (r+1)/\sqrt{N}]\}$. From our construction, there exists a constant

$C_3 = C_3(\sigma, \delta)$ such that $J_i(r) \leq C_3 r$ for any $i \in \{1, \dots, N\}$ and all $r < \epsilon\sqrt{N}$, if ϵ is smaller than some ϵ_0 independent of δ. Thus, applying Proposition 10, we have that

$$N^{-2} \sum_{i \neq j, d(Z_i, Z_j) \leq \epsilon} |G_h(Z_i, Z_j)| \leq \frac{C_3(\epsilon, \delta, \sigma)}{N^2} \sum_{i=1}^{N} \sum_{k=1}^{\epsilon\sqrt{N}} k \log(k/\sqrt{N}) = O(\frac{\log N}{\sqrt{N}}) .$$

For $\epsilon' > 0$ given, fix $\epsilon > 0$ so that

$$|\mathcal{E}_h(\sigma) - \mathcal{E}_h^{\epsilon}(\sigma)| < \epsilon' .$$

Then, taking $N \to \infty$ we conclude that

$$\limsup_{N\to\infty} |N^{-2} \sum_{i \neq j} G_h(Z_i, Z_j) - \mathcal{E}_h(\sigma)| \leq C_2(\epsilon, \delta) + \epsilon' .$$

In particular, taking $\delta = \delta(\epsilon')$ small enough gives

$$\limsup_{N\to\infty} |N^{-2} \sum_{i \neq j} G_h(Z_i, Z_j) - \mathcal{E}_h(\sigma)| \leq 2\epsilon' .$$

By Lemma 26, reducing δ further if necessary, we also have $|J^{h,K}(\sigma_N) - J^{h,K}(\sigma)| \leq \epsilon'$. Combining these estimates and substituting in (61), one gets that for any $\epsilon' > 0$ and all N large enough,

$$\mathbf{Prob}_N(B(\sigma, \delta)) \geq e^{-N^2 I(\sigma) - 3\epsilon' N^2} \int_{D_I^\eta} \kappa_N \tag{62}$$

To complete the proof, we again use (59). Indeed, as above (see Proposition 5), κ is a smooth positive $(1,1)$ form on $\mathbb{CP}^1$. Now, $D_I^\eta \subset (\mathbb{CP}^1)^N$ is a product of the one-complex dimensional sets $D_j^{\eta;N} := \{\zeta_j : d(\zeta_j, Z_j) \leq \frac{\eta}{N}\} \subset \mathbb{CP}^1$. Hence,

$$\int_{D_I^\eta} \kappa_N = \left(\int_{D_1^{\eta;N}} \kappa \right)^N .$$

Since κ is a smooth area form, $\int_{D_1^{\eta;N}} \kappa \sim CN^{-2}$. It follows that

$$\frac{1}{N^2} \log \int_{D_I^\eta} \kappa_N = O(\frac{\log N}{N}).$$

Combined with (62) and the fact that ϵ' was arbitrary, this completes the proof. □

7.4. **The normalizing constant: Proof of Lemma 4.** Finally, we consider the normalizing constants of Proposition 3, in particular the determinant $\det \mathcal{A}_N(h, \nu) = \det \left(\langle z^k, \psi_\ell \rangle \right)$ of the change of basis matrix (32). Here, $\langle \cdot, \cdot \rangle$ is the inner product $G_N(h, \nu)$. The same asymptotics have been studied before in the theory of orthogonal polynomials (see e.g. [B]) and in the setting of line bundles in [BB]. The following gives an alternative to the proof in [BB] in the special case at hand.

We claim that

$$\lim_{N\to\infty} \frac{1}{N^2} \log \hat{Z}_N(h) = \frac{1}{2} \log \mathrm{Cap}_h(K).$$

Proof. We prove this by combining the large deviations result for the un-normalized probability measure with the fact that $\mathbf{Prob}_N$ is a probability measure. By Lemma 29 and proof of the large deviations upper bound,

$$\begin{aligned} 0 &= \lim_{N\to\infty} \tfrac{1}{N^2} \log \mathbf{Prob}_N(\mathcal{M}(\mathbb{CP}^1)) \\ &\leq \limsup_{N\to\infty} \tfrac{-1}{N^2} \log \hat{Z}_N(h) - \inf_{\mu\in\mathcal{M}(\mathbb{CP}^1)} I^{h,K}(\mu) \\ &= \limsup_{N\to\infty} \tfrac{-1}{N^2} \log \hat{Z}_N(h) - I^{h,K}(\nu_{h,K}) \\ &= \limsup_{N\to\infty} \tfrac{-1}{N^2} \log \hat{Z}_N(h) - \tfrac{1}{2} \log \mathrm{Cap}_h(K). \end{aligned}$$

A similar argument using the large deviations lower bound shows the reverse inequality for $\liminf_{N\to\infty} \frac{-1}{N^2} \log \hat{Z}_N$. □

Corollary **32.**

$$\lim_{N\to\infty} \frac{1}{N^2} \log |\det \mathcal{A}_N(h)|^{-2} = -\frac{1}{2} E(h) + \frac{1}{2} \log \mathrm{Cap}_h(K).$$

Proof. By Lemma 17,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \hat{Z}_N(h) = \lim_{N\to\infty} \frac{1}{N^2} \log |\det \mathcal{A}_N(h)|^{-2} + \frac{1}{2} E(h).$$

□

8. Appendix

This Appendix contains proofs of some technicalities used in the proofs of Theorem 1.

8.1. **Regularization of measures.** In the large deviations lower bound, we will need to prove that any $\mu \in \mathcal{M}(\mathbb{CP}^1)$ may be weakly approximated by measures with continuous densities. In [BZ], this was proved using convolution with Gaussians; since we are working on $\mathbb{CP}^1$, we need a suitable replacement.

We once again use the auxiliary Kähler metric and its Laplacian Δ_ω. It generates the heat operator $e^{t\Delta_\omega}$. We denote its heat kernel by

$$K_\omega(t, z, w) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \varphi_j(z) \varphi_j(w),$$

and write

$$e^{t\Delta_\omega} \mu(z) = \int_{\mathbb{CP}^1} K_\omega(t, z, w) d\mu(w).$$

It is well-known and easy to see that $e^{t\Delta_\omega}\mu(z) \in C^\infty(\mathbb{CP}^1)$ for any $\mu \in \mathcal{M}(\mathbb{CP}^1)$. The following simple lemma is sufficient for our purposes. Recall the energy norm $\|\cdot\|_\omega$ on $\mathcal{M}(\mathbb{CP}^1)$, see (47).

Lemma **33.** *If $I^{h,K}(\mu) < \infty$, then $\|e^{t\Delta_\omega}\mu\|_\omega \to \|\mu\|_\omega$ as $t \to 0^+$. In particular, $I^{h,K}(e^{t\Delta_\omega}\mu) \to I^{h,K}(\mu)$. Moreover, $e^{t\Delta_\omega}\mu \in \mathcal{M}(\mathbb{CP}^1)$.*

Proof. It is well known (and follows by the maximum principle for the heat equation) that $K_\omega(t,z,w) > 0$. Hence, $\mu_t := \left(e^{t\Delta_\omega}\mu\right)\omega$ is a positive measure. Further, $\int_{\mathbb{CP}^1}\mu_t = 1$ since $\int_{\mathbb{CP}^1} K_\omega(t,z,w)\omega = 1$.

The claimed convergence in energy norm is equivalent to the statement $e^{t\Delta_\omega}\mu \to \mu$ in $H^{-1}(\mathbb{CP}^1)$. To see the latter, it suffices to observe that by monotone convergence,

$$\lim_{t\to 0^+}\sum_{j=1}^{\infty}\frac{1-e^{t\lambda_j}}{\lambda_j}|\mu(\varphi_j)|^2 = 0.$$

□

8.2. **Residue at infinity and the evaluation of certain integrals.** The purpose of this section is to define the Robin constant $\rho_\varphi(\infty)$ in (7), to go over the calculation of $E(h)$ in (7) and in Lemma 8, and to prove (ii) of Lemma 9. The calculations involve the 'residue at infinity' of the integrals:

- (i) $\int_{\mathbb{C}} 2\log|z-w| dd^c\varphi = \varphi(z) - 4\pi\rho_\varphi(\infty)$ of Lemma 8 (see (38)). Here, $e^{-\varphi}$ is the local expression on $\mathbb{C}$ of a global smooth Hermitian metric h_φ on $\mathcal{O}(1)$.
- $\int_{\mathbb{C}}\varphi dd^c\varphi$ (with the same notation as (i)).
- (ii) $\int_{\mathbb{C}}\log||s_\zeta(w)||^2_{h^N} dd^c\varphi = N(\int\varphi d\mu_\zeta - E(h))$ of Lemma 9, where $s_\zeta(z) = \prod_{j=1}^N (z-\zeta_j)e^N(z)$.

These integrals are of the form $\int_{\mathbb{C}} v dd^c u$ for special u, v. We calculate them by integration by parts. The subtlety is the 'boundary term' or 'residue' at infinity. To illustrate, we note that in the case of the Fubini-Study Hermitian metric h_{FS} on $\mathcal{O}(1)\to\mathbb{CP}^1$, the Chern form $dd^c\log(1+|z|^2)^{1/2}$ is exact on $\mathbb{C}$, but its integral $\int_{\mathbb{C}} dd^c\log(1+|z|^2)^{1/2} = 1$ and not zero. In the terminology of [BT], the Robin constant ρ_u of a subharmonic function u on $\mathbb{C}$ is defined by

$$\rho_u(z) = \limsup_{\lambda\to\infty}\left(u(\lambda z) - \log^+|\lambda z|\right). \tag{63}$$

We denote by $\mathcal{L}$ the Lelong class of subharmonic functions satisfying $u(z)\le\log^+|z| + C$, and put $\mathcal{L}_\rho := \{u\in\mathcal{L}(\mathbb{C}) : \rho_u \ne -\infty\}$. Note that $\rho_1 \equiv -\infty$. It is proved in [BT] that if $u, v\in\mathcal{L}_\rho$, then

$$\int_{\mathbb{C}} u dd^c v - v dd^c u = 2\pi(\rho_u(\infty) - \rho_v(\infty)). \tag{64}$$

LEMMA **34.** *Let h be a smooth Hermitian metric on $\mathcal{O}(1)$ and let $h = e^{-\varphi}$ in the frame e_1 over the affine chart $U_1 = \mathbb{C}$. Then,*

$$\int\log|z-w| dd^c\varphi = \frac{\varphi(z)}{2} - 2\pi\rho_\varphi(\infty)\,, \tag{65}$$

$$\int_{\mathbb{C}}\log||s_\zeta(w)||^2_{h^N} dd^c\varphi = N(\int\varphi d\mu_\zeta - E(h))\,. \tag{66}$$

Proof. For the integral in (65), we first consider the Fubini-Study case, where the Hermitian metric h_{FS} is locally given in the standard frame by $\varphi(w) = \log(1+|w|^2)^{1/2}$. It is simple to see that

$$\int_{\mathbb{C}}\log|z-w| dd^c\log(1+|w|^2) = \tfrac{1}{2}\log(1+|z|^2).$$

This follows from (26), (64) and the fact that $\rho_{\frac{1}{2}\log(1+|w|^2)}(\infty) = \rho_{\log|z-w|}(\infty) = 0$, since for sufficiently large $|w|$,

$$\log|z-\lambda w| - \log|\lambda w| = \log|1-\tfrac{z}{\lambda w}| = \Re\log(1-\tfrac{z}{\lambda w}) = -\Re\tfrac{z}{\lambda w} + \cdots = o(1),$$

$$\log(1+|\lambda w|^2)^{1/2} - \log|\lambda w| = \log(1+|\lambda w|^{-1})^{1/2} = o(1).$$

In the case of a general Hermitian metric h on $\mathcal{O}(1)$, we may write $h = h_{FS}e^{-\Phi}$ where $\Phi \in C^\infty(\mathbb{CP}^1)$. Then on $\mathbb{C}$, the weight has the form $\varphi(w) = \log(1+|w|^2)^{1/2} + \Phi(w) \in \mathcal{L}_\rho$ with $\rho_\varphi(\infty) = \Phi(\infty)$. By (64) we have,

$$\int_{\mathbb{C}} \log|z-w| dd^c\varphi \;=\; (\tfrac{1}{2}\varphi(z) - 2\pi\rho_\varphi(\infty)) = \tfrac{1}{2}\varphi(z) - 2\pi\Phi(\infty), \tag{67}$$

proving (65).

For the integral in (66), we again express the Hermitian metric as $h = e^{-\Phi}h_{FS}$. We then use (66) to obtain,

$$\begin{aligned}
\int_{\mathbb{CP}^1} \log||s_\zeta||^2_{h^N}\omega_h &= 2\sum_{j=1}^N \int_{\mathbb{C}} \log|z-\zeta_j| dd^c(\log(1+|z|^2)^{\frac{1}{2}} + \Phi(z)) - N\int_{\mathbb{C}} \varphi\omega_h \\
&= \sum_j (\log(1+|\zeta_j|^2)^{\frac{1}{2}} + \Phi(\zeta_j) - 4\pi\Phi(\infty)) - N\int_{\mathbb{C}} \varphi\omega_h \\
&= N\int (\log(1+|w|^2)^{\frac{1}{2}} + \Phi(w) - 4\pi\Phi(\infty)) d\mu_\zeta(w) - N\int_{\mathbb{C}} \varphi\omega_h \\
&= N(\int \varphi d\mu_\zeta - \big(\int_{\mathbb{CP}^1} \varphi(z)\omega_h + 4\pi\rho_\varphi(\infty)\big)),
\end{aligned}$$

as claimed.

□

9. Notational appendix

(1) $G_h(z,w)$: Green's function w.r.t. h (see (5)).
(2) $E(h)$: see (7).
(3) $U^\mu_h(z)$: Green's potential (see (8)).
(4) $\mathcal{E}_h(\mu)$: Green's energy (see (9)).
(5) Cap_h: Green's capacity (see (51)).
(6) $E_0(h) = \frac{1}{2}\log\mathrm{Cap}_h(K)$: (see (15)).
(7) $\rho_\varphi(\infty)$: Robin constant (see (63) of §8.2, and also (38)).
(8) $I^{h,K}(\mu)$: rate function (see (14)).
(9) $\nu_{h,K}$: equilibrium measure w.r.t. (h,K) (see Proposition 19).
(10) $Z_N(h), \hat{Z}_N(h)$: normalizing constants (see (25)).
(11) $s_\zeta(z) = \prod_{j=1}^N (z-\zeta_j)e^N(z)$ (see (42)).

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