Slowdown for time inhomogeneous branching Brownian motion

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Abstract

We consider the maximal displacement of one dimensional branching Brownian motion with (macroscopically) time varying profiles. For monotone decreasing variances, we show that the correction from linear displacement is not logarithmic but rather proportional to $T^{1/3}$. We conjecture that this is the worse case correction possible.

1 Introduction and statement of results

The classical branching Brownian motion (BBM) model in $\mathbb{R}$ can be described probabilistically as follows. At time $t = 0$, one particle exists and is located at the origin. This particle starts performing Brownian motion, up to an exponentially distributed random time. At that time, the particle instantaneously splits into two independent particles, and those start afresh performing Brownian motion until their (independent) exponential clock rings.

We introduce some notation. Let $\mathcal{N}_t$ denote the collection of particles alive at time t, set $N(t) = |\mathcal{N}_t|$, and for any particle $v \in \mathcal{N}_t$, let $x_v(s), s \in [0, t]$ denote the (Brownian) trajectory performed by the particle and its ancestors. $N(t)$ is a continuous time branching process, and it is straightforward to verify that $N(t)e^{-t}$ is a Martingale, which converges almost surely to a positive, finite random variable n_∞. In particular, $t^{-1} \log N(t)$ converges almost surely to 1.

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We will be interested in the location of the maximal particle, i.e. in the random variable

$$M_t = \max_{v \in \mathcal{N}_t} x_v(t)\,.$$

As is well known, the distribution function $F(x,t) = P(M_t \geq x)$ satisfies the Kolmogorov-Petrovskii-Piskunov equation (also attributed to Fisher)

$$\frac{\partial F}{\partial t}(x,t) = \frac{1}{2}\frac{\partial^2 F}{\partial^2 x}(x,t) + F(x,t)(1-F(x,t))\,, \quad F(x,0) = \mathbf{1}_{x \leq 0}\,.$$

See [M75] for a probabilistic interpretation of the KPP equation.

In a seminal work, Bramson [Br78] showed among other facts that

$$M_t = m(t) + O_P(1)\,, \quad m(t) = \sqrt{2}t - \frac{3}{2}\cdot\frac{1}{\sqrt{2}}\log t\,, \tag{1}$$

in the sense that for any ϵ there is a K_ϵ so that

$$P(|M_t - m(t)| > K_\epsilon) \leq \epsilon\,.$$

In particular, $\mathrm{Med}(M_t) = m(t) + O(1)$, where $\mathrm{Med}(M_t)$ denotes the median of M_t. (In subsequent work [Br83], Bramson also discusses convergence to a shifted traveling wave, but this is not the focus of the current work.) Analogues of (1) also hold in the setup of discrete time branching random walks, see [ABR09]; for a recent convergence result for BRWs, see [Ai11].

The leading term in (1), linear in time, is a relatively straight–forward consequence of large deviations computations and the first and second moment methods; in particular, the coefficient $\sqrt{2}$ would be the same if instead of BBM, one would consider the maximum of e^t independent Brownian motions run for time t. On the other hand, the logarithmic correction term in (1) is more subtle, and reflects the correlation structure of the BBM: for the maxima of independent BMs, the $3/2$ multiplying the logarithmic term is replaced by $1/2$. For a pedestrian introduction to these issues, see the lecture notes [Z12].

Our goal in this paper is to address situations in which the diffusivity of the Brownian motion changes in time, in a macroscopic scale. This is motivated in part by our earlier work [FZ11], in which we showed that the corrrection factor $3/2$ multiplying the logarithmic term can be replaced by different, and eventually much larger, factors. This naturally leads to the question, whether larger-than-logarithmic corrections are possible. Our goal here is to answer this question in the affirmative.

We turn to the description of the time inhomogeneous BBM that we consider. Fix T (eventually, large). We consider the BBM model where at time t, all particles move independently as Brownian motions with variance $\sigma_T^2(t) = \sigma^2(t/T)$, and branch independently at rate 1. Here, σ is a smooth, strictly decreasing function on $[0,1]$ with range in a compact subset of $(0,\infty)$, whose derivative is bounded above by a strictly negative constant. Define $\mathcal{N}_t$, $\{x_v(t)\}_{v\in\mathcal{N}_t}$ and M_t, $t\in[0,T]$, as in the case of time homogeneous BBM. This model has been considered before in [DS88]. Our main result is the following.

Theorem 1.1. *With notation as above, we have that*

$$Med(M_T) = v_\sigma T - g_\sigma(T)\,, \tag{2}$$

where v_σ is defined in (6)*, and*

$$0 < \liminf_{T\to\infty} \frac{g_\sigma(T)}{T^{1/3}} \leq \limsup_{T\to\infty} \frac{g_\sigma(T)}{T^{1/3}} < \infty \tag{3}$$

We emphasize that it is already known a-priori [Fa10] that $\{M_T - \text{Med}(M_T)\}$ is a tight sequence; in fact, the tails estimates in [Fa10] are strong enough to allow one to replace, both in the statement above and in Theorem 1.1, the median $\text{Med}(M_T)$ by EM_T.

2 Proofs

Before bringing the proof of Theorem 1.1, we collect some preliminary information concerning the path of individual particles. With $W.$ and $\tilde{W}.$ denoting standard Brownian motions, let

$$X_t = \int_0^t \sigma_T(s)dW_s\,, \quad t\in[0,T].$$

Let $\tau(t) = \int_0^t \sigma_T^2(s)ds$. Clearly, $X.$ has the same law as $\tilde{W}_{\tau(t)}$. The following is a standard adaptation of Schilder's theorem [S66, DZ98], using the scaling properties of Brownian motion.

Theorem 2.1 (Schilder)**.** *Define $Z_t = \frac{1}{T}X_{t/T}, t\in[0,1]$. Then Z_t satisfies a large deviation principle in $C_0[0,1]$ of speed T and rate function*

$$I(f) = \begin{cases} \int_0^1 \frac{f'(s)^2}{2\sigma^2(s)}ds\,, & f\in H_1[0,1]\,, \\ \infty\,, & \text{else} \end{cases}\,.$$

Here, $H_1[0,1]$ is the space of absolutely continuous function on $[0,1]$ that vanish at 0, whose (almost everywhere defined) derivative is square-integrable.

We now wish to define a barrier for the particle systems that is unlikely to be crossed. This barrier will also serve as a natural candidate for a change of measure. Recall that at time t, with overwhelming probability there are at most $e^{t+o(t)}$ particles alive in the system. Thus, it becomes unlikely that any particle crosses a boundary of the form $Tf(\cdot/T)$ if, at any time,

$$J_t(f) := \int_0^t \frac{f'(s)^2}{2\sigma^2(s)} ds > t\,.$$

This motivates the following lemma.

Lemma 2.2. *Assume σ is strictly decreasing. Then the solution of the variational problem*

$$v_\sigma := \sup\{f(1) : J_t(f) \le t, t \in [0,1]\} \tag{4}$$

exists, and the unique minimizing path is the function

$$\bar{f}(t) = \sqrt{2} \int_0^t \sigma(s) ds\,. \tag{5}$$

In particular,

$$v_\sigma = \sqrt{2} \int_0^1 \sigma(s) ds\,. \tag{6}$$

Proof of Lemma 2.2: We are going to prove that no other functions can do better than $\bar{f}$. That is, if some absolutely continuous function g satisfies $g(0) = 0$ and the constraint $J_t(g) \le t$ for all $0 \le t \le 1$, then $g(1) \le \bar{f}(1) = v_\sigma$. In fact, denote $\phi(t) = J_t(g) \le t$ for $0 \le t \le 1$, and then $\phi'(t) = \frac{g'(t)^2}{2\sigma^2(t)}$ a.e.. We can write $g^2(1)$ as

$$g^2(1) \;=\; \left(\int_0^1 g'(t)dt\right)^2 = \left(\int_0^1 \sqrt{2\phi'(t)}\sigma(t)dt\right)^2.$$

Using Hölder's inequality, we have

$$g^2(1) \le 2\left(\int_0^1 \phi'(t)\sigma(t)dt\right)\left(\int_0^1 \sigma(t)dt\right) = \sqrt{2}v_\sigma \left(\int_0^1 \phi'(t)\sigma(t)dt\right).$$

Using integration by parts, the above is equal to

$$\sqrt{2}v_\sigma\left(\phi(1)\sigma(1) - \int_0^1 \phi(t)\sigma'(t)dt\right).$$

Since $\phi(t) \le t$ and $\sigma'(t) \le 0$ for all $0 \le t \le 1$, the above is less than or equal to

$$\sqrt{2}v_\sigma\left(\sigma(1) - \int_0^1 t\sigma'(t)dt\right) = \sqrt{2}v_\sigma \int_0^1 \sigma(t)dt = v_\sigma^2,$$

where we apply integration by parts in the first equality. This completes the proof. □

Proof of Theorem 1.1: We begin with the upper bound in (2). The first step is to show that in fact, no particle will be found significantly above $T\bar{f}(t/T)$.

Lemma 2.3. *There exists C large enough such that, with*

$$\mathcal{A} = \{\exists t \in [0,T], v \in \mathcal{N}_t, x_v(t) > T\bar{f}(t/T) + C\log T\},$$

it holds that

$$P(\mathcal{A}) \to_{T\to\infty} 0\,. \tag{7}$$

Proof of Lemma 2.3: Recall the process $X_\cdot$ in $C_0[0,T]$, whose law we denote by P_0. Consider the change of measure with Radon–Nykodim derivative

$$\begin{aligned} \frac{dP_1}{dP_0}|_{\mathcal{F}_t} &= \exp\left(-\int_0^t \frac{\bar{f}'(s/T)}{\sigma^2(s/T)}dX_s - \frac{1}{2}\int_0^t \frac{(\bar{f}'(s/T))^2}{\sigma^2(s/T)}ds\right) \\ &= \exp\left(-\int_0^t \frac{\sqrt{2}}{\sigma(s/T)}dX_s - t\right). \end{aligned} \tag{8}$$

The process $X_\cdot$ under P_0 is the same as the process $X_\cdot + T\bar{f}(\cdot/T)$ under P_1. Note that for any $t \le T$,

$$\int_0^t \frac{\sqrt{2}}{\sigma(s/T)}dX_s = \frac{\sqrt{2}X_t}{\sigma(t/T)} + \frac{\sqrt{2}}{T}\int_0^t X_s \frac{\sigma'(s/T)}{\sigma^2(s/T)}ds\,. \tag{9}$$

We then have, with $\tau = \inf\{t \le T : X_t \ge C\log T\}$, on the event $\tau \le T$,

$$\begin{aligned} \int_0^\tau \frac{(\bar{f}'(s/T))}{\sigma^2(s/T)}dX_s &\ge \frac{\sqrt{2}C\log T}{\sigma(t/T)} + \frac{\sqrt{2}C\log T}{T}\int_0^t \frac{\sigma'(s/T)}{\sigma^2(s/T)}ds \\ &= \frac{\sqrt{2}C\log T}{\sigma(0)}, \end{aligned}$$

and therefore, with $\tau' = \inf\{t \le T : X_t \ge T\bar{f}(t/T) + C\log T\}$, we have, for $k \le T$,

$$\begin{aligned} P_0(\tau' \in [k-1,k)) &= P_1(\tau \in [k-1,k)) = E_{P_0}\left(\frac{dP_1}{dP_0}\mathbf{1}_{\tau\in[k-1,k)}\right) \\ &\le E_{P_0}\left(\mathbf{1}_{\tau\in[k-1,k)}\exp\left(-\frac{\sqrt{2}C\log T}{\sigma(0)} - \tau\right)\right). \end{aligned}$$

Define

$$\theta = \inf\{t \le T : \text{there is } v \in \mathcal{N}_T \text{ so that } x_v(t) \ge T\bar{f}(t/T) + C\log T\},$$

and Z_k to be the number of particles $z \in \mathcal{N}_k$ such that $x_v(t) \leq T\bar{f}(t/T) + C\log T$ for all $t \leq k-1$ and $x_v(t) \geq T\bar{f}(t/T) + C\log T$ for some $k-1 \leq t \leq k$. Then,

$$P(\theta \leq T) \leq \sum_{k=1}^{T} P(\theta \in [k-1,k)) \leq P(Z_k \geq 1)\,,$$

and, using a first moment computation, we obtain

$$P(Z_k \geq 1) \leq EZ_k \leq e^k P_0(\tau' \in [k-1,k)) \leq \exp\left(-\frac{\sqrt{2}C\log T}{\sigma(0)} + 1\right).$$

Therefore,

$$P(\theta \leq T) \leq T\exp\left(-\frac{\sqrt{2}C\log T}{\sigma(0)} + 1\right).$$

This completes the proof of Lemma 2.3. $\square$

We need one more technical estimate.

Lemma 2.4. *With $X_\cdot$ and C as in Lemma 2.3, there exists a constant $C' \in (0,1)$ so that*

$$e^T P_0(X_t \leq T\bar{f}(t/T) + C\log T, t \in [0,T], X_T \geq T\bar{f}(1) - C'T^{1/3}) \to_{T\to\infty} 0\,. \tag{10}$$

Proof of Lemma 2.4: Fix $C' \in (0,1)$. We apply a change of measure similar to the one used in Lemma 2.4, whose notation we continue to use. We deduce the existence of positive constants c_1, c_2 (independent of T) such that

$$\begin{aligned} & P_0(X_t \leq T\bar{f}(t/T) + C\log T, t \in [0,T], X_T \geq T\bar{f}(1) - C'T^{1/3}) \\ \leq\ & e^{-T} e^{c_1(C'T^{1/3} + \log T)} \\ & \cdot E_{P_0}\left(\exp\left(\frac{c_2}{T}\int_0^T X_s ds\right) \mathbf{1}_{X_T \geq -C'T^{1/3}} \mathbf{1}_{X_t \leq 0, t \leq T}\right), \end{aligned}$$

where here we used that $-\sigma'$ is bounded below by a positive constant and σ is bounded above. By representing $X_\cdot$ as a time-changed Brownian motion, the lemma will follows (for a small enough C') if we can show that for any constant c_3 there exists a $c_4 = c_4(c_3) > 0$ independent of $C' \in (0,1)$ such that

$$\mathtt{D} := E\left(\exp\left(\frac{c_3}{T}\int_0^T B_s ds\right) \mathbf{1}_{B_T \geq -C'T^{1/3}} \mathbf{1}_{B_t \leq 0, t \leq T}\right) \leq e^{-c_4 T^{1/3}}\,, \tag{11}$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion started at $-C\log T$. Note however that

$$\mathtt{D} \leq E\left(\exp\left(-\frac{c_3}{T}\int_0^T |B_s|ds\right)\mathbf{1}_{|B_T|\leq T^{1/3}}\right)e^{c_5\log T} \leq e^{-c_4T^{1/3}}\,,$$

where here $B.$ is a Brownian motion started at 0 and the last inequality is a consequence of known estimates for Brownian motion, see e.g. [BS96, Formula 1.1.8.7, pg. 141]. □

Remark The estimate in (11) can also be derived probabilistically. Here is a sketch. It is clearly enough to estimate the expectation on the event $\mathtt{F} := \left\{\mathrm{Leb}(\{s : 0 \geq B_s \geq -c_5T^{1/3}\}) \geq 1/2\right\}$. But $P(\mathtt{F})$ decays exponentially in $T^{1/3}$. Some more details are provided in [Z12].

We have completed all steps required for the proof of the upper bound in Theorem 1.1. Due to the strong tightness result in [Fa10] and Lemma 2.3, it is enough to show that

$$P(\{M_T \geq \bar{T}f(1) - C'T^{1/3}\} \cap \mathcal{A}^{\mathtt{C}}) \to 0\,.$$

This follows from the first moment method and Lemma 2.4.

We turn to the proof of the lower bound. Call a particle $v \in \mathcal{N}_T$ *good* if

$$T\bar{f}(t/T) - T^{1/3} \leq x_v(t) \leq T\bar{f}(t/T),\ \text{for all } t \leq T.$$

Set

$$\mathcal{M} = \sum_{v\in\mathcal{N}_T} \mathbf{1}_{v \text{ is a good particle}}\,.$$

Lemma 2.5. *There exists a constant $C > 0$ such that*

$$P(\mathcal{M} \geq 1) \geq e^{-CT^{1/3}}.$$

Proof of Lemma 2.5: Recall the process $X.$ in $C_0[0, T]$, whose law we denoted by P_0, and the measure P_1 defined by (8). We then calculate the first moment

$$\begin{aligned}
E\mathcal{M} &= E\sum_{v\in\mathcal{N}(T)} \mathbf{1}_{v \text{ is a good particle}}\\
&= e^T P_0(T\bar{f}(t/T) - T^{1/3} \leq X_t \leq T\bar{f}(t/T),\ \text{for all } t \leq T)\\
&= E_{P_0}\left[\exp\left(-\int_0^T \frac{\sqrt{2}}{\sigma(s/T)}dX_s\right)\mathbf{1}_{\{-T^{1/3}\leq X_t\leq 0,\ \text{for all } t\leq T\}}\right].
\end{aligned}$$

Repeating the computation in (9), we conclude that

$$E\mathcal{M} \geq \exp\left(-c_6T^{1/3}\right)P_0\left(-T^{1/3} \leq X_t \leq 0,\ \text{for all } t \leq T\right).$$

Since under P_0, $X_\cdot$ is a time changed Brownian motion, we have that

$$P_0\left(-T^{1/3} \le X_t \le 0, \text{ for all } t \le T\right) \ge e^{-c_7 T^{1/3}}$$

for some $c_7 > 0$. Hence,

$$E\mathcal{M} \ge e^{-c_8 T^{1/3}}$$

for some $c_8 > 0$.

We next derive an upper bound for the second moment $E\mathcal{M}^2$. By definition,

$$E\mathcal{M}^2 = E \sum_{v,v' \in \mathcal{N}_T} \mathbf{1}_{v,\, v' \text{ are good particle}}.$$

When $v \neq v'$, we let $t_{vv'}$ be the branching time of the last common ancestor of v and v'. Then, the paths $\{x_v(s)\}_{0\le s\le T}$ and $\{x_{v'}(s) - x_{v'}(t_{vv'})\}_{t_{vv'}\le s\le T}$ are independent. Applying a change of measure similar to that used in the computation of $E\mathcal{M}$, we can bound above

$$\begin{aligned}
&E\mathcal{M}^2 \le E\mathcal{M} \\
&+ \int_0^T e^{2T-t} E\left[\exp\left(-\int_0^T \frac{\sqrt{2}}{\sigma(s/T)} dX_s^1 - T\right) \mathbf{1}_{\{-T^{1/3}\le X_s^1 \le 0, \text{ for all } 0\le s\le T\}}\right. \\
&\quad \left.\cdot \exp\left(-\int_t^T \frac{\sqrt{2}}{\sigma(s/T)} dX_s^2 - (T-t)\right) \mathbf{1}_{\{-T^{1/3}\le X_s^2 \le T^{1/3}, \text{ for all } t\le s\le T\}}\right] dt,
\end{aligned}$$

where $X_\cdot^1$ and $X_\cdot^2$ are two i.i.d. copies of $X_\cdot$ (under the law P_0). The above is equal to

$$\begin{aligned}
&E\mathcal{M} + \int_0^T E\left[\exp\left(-\int_0^T \frac{\sqrt{2}}{\sigma(s/T)} dX_s^1\right) \mathbf{1}_{\{-T^{1/3}\le X_s^1 \le 0, \text{ for all } 0\le s\le T\}}\right. \\
&\quad \left.\cdot \exp\left(-\int_t^T \frac{\sqrt{2}}{\sigma(s/T)} dX_s^2\right) \mathbf{1}_{\{-T^{1/3}\le X_s^2 \le T^{1/3}, \text{ for all } t\le s\le T\}}\right] dt.
\end{aligned}$$

On the event $\{-T^{1/3} \le X_s^1 \le 0, \text{ for all } 0 \le s \le T\}$, using integration by parts, one has

$$-\int_0^T \frac{1}{\sigma(s/T)} dX_s^1 = -\frac{X_T^1}{\sigma(1)} - \frac{1}{T}\int_0^T \frac{X_s^1 \sigma'(s/T)}{\sigma^2(s/T)} ds \le \frac{T^{1/3}}{\sigma(1)}.$$

Similarly, on the event $\{-T^{1/3} \le X_s^2 \le T^{1/3}, \text{ for all } t \le s \le T\}$, using integration by parts, one has

$$\begin{aligned}
-\int_t^T \frac{1}{\sigma(s/T)} dX_s^2 &= -\frac{X_T^2}{\sigma(1)} + \frac{X_t^2}{\sigma(t/T)} - \frac{1}{T}\int_t^T \frac{X_s^2 \sigma'(s/T)}{\sigma^2(s/T)} ds \\
&\le \frac{3T^{1/3}}{\sigma(1)}.
\end{aligned}$$

Therefore, $E\mathcal{M}^2$ is bounded above by $E\mathcal{M}$ plus

$$\int_0^T e^{4T^{1/3}/\sigma(1)}$$
$$E\left[\mathbf{1}_{\{-T^{1/3}\le X_s^1\le 0,\text{ for all } 0\le s\le T\}}\mathbf{1}_{\{-T^{1/3}\le X_s^2\le T^{1/3},\text{ for all } t\le s\le T\}}\right] dt\,.$$

The latter integral is less than or equal to

$$\int_0^T e^{4T^{1/3}/\sigma(1)} dt \le e^{c_9 T^{1/3}}$$

for some $c_9 > 0$. Hence, one can apply the second moment method:

$$P(\mathcal{M}\ge 1) = P(\mathcal{M}>0) \ge \frac{(E\mathcal{M})^2}{E\mathcal{M}^2} \ge e^{-c_{10}T^{1/3}}.$$

This completes the proof of the lemma by letting $C = c_{10} > 0$. $\square$

By a direct first moment computation (or using the already proved upper bound (3)), the minimum position of particles at time $AT^{1/3}$ is, with high probability, greater than or equal to $-A'T^{1/3}$ for a constant A' depending on A. Choosing $A > C$ (with C as in Lemma 2.5), and using the independence of the motion of descendents of different particles in $\mathcal{N}_{AT^{1/3}}$, the lower bound in (3) follows. $\square$

3 Discussion

Our choice of considering strictly decreasing diffusivity is not accidental: the computations in [FZ11] hint that this should correspond to a worse-case situation. In fact, we conjecture the following.

Conjecture 3.1. *Let σ be a smooth function from $[0,1]$ to a compact subset of $(0,\infty)$. Then, (2) holds with v_σ determined by (4) and $|g_\sigma(T)| = O(T^{1/3})$.*

In a slightly more technical direction, it would of course be of interest, in the setup of Theorem 1.1, to show that $g_\sigma(T)/T^{1/3}$ converges as $T\to\infty$, and to evaluate the limit. Our methods are not refined enough to allow for that.

Finally, we mention that results for the homogenization of the KPP equation are available, see e.g. [N11, NRRZ12] and references therein. In the terminology we employ here, those results correspond to fast varying, or *microscopic*, time inhomogenuities.

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