Homework Set 5

Let $1 \leq p < \infty$ and consider the space \( \text{weak-}L^p(X, \mu) \), which we defined to be the set of measurable functions on \( X \) such that
\[
[f]_{w,p} := \sup_{\alpha > 0} \alpha [\lambda_f(\alpha)]^{1/p} < \infty,
\]
where \( \lambda_f(\alpha) := \mu\{x : |f(x)| > \alpha\} \). As we saw, if \( f \in L^p \), then \( [f]_{w,p} \leq \|f\|_p \). Hence \( L^p \subset \text{weak-}L^p \).

1. (a) Show that \( \text{weak-}L^p \) is a linear space.
(b) Show that for all \( \beta \in \mathbb{C} \), we have \( [\beta f]_{w,p} = |\beta|[f]_{w,p} \).
(c) Show that \( [f]_{w,p} = 0 \) if and only if \( f = 0 \) \( \mu \)-a.e. in \( X \).
(d) Show that the triangle inequality can fail for \([\cdot]_{w,p}\) for all \( p \). Hence \([\cdot]_{w,p}\) is not a norm in general. (Hint: It happens even when \( X \) consists of just 2 points.)
(e) A quasi-norm \([\cdot]\) satisfies all the properties of a norm except the triangle inequality is replaced by \([f + g] \leq C([f] + [g]) \). Show that \([\cdot]_{w,p}\) is a quasi-norm with \( C \leq 2 \).

2. (see also: Lieb and Loss, p. 121, ex. 1 and 2.) Let \( (X, \mu) \) be \( \sigma \)-finite. For \( p > 1 \), define
\[
\|f\|_{w,p} := \sup_A \frac{1}{\mu(A)^{1/p'}} \int_A |f|d\mu,
\]
where the supremum is taken over all subsets \( A \) of \( X \) with \( \mu(A) < \infty \), and \( 1/p + 1/p' = 1 \).
(a) Show that \( \|\cdot\|_{w,p} \) is a norm.
(b) Show that there exists a constant \( C < \infty \), depending only on \( p \), such that
\[
[f]_{w,p} \leq \|f\|_{w,p} \leq C[f]_{w,p}
\]
for all \( f \).
(Hint: The first inequality is easy. For the second inequality, you have to solve an optimization problem: Using the “layer-cake representation”, write
\[
\int_A |f|d\mu = \int_0^\infty \mu(\{|f| > t\} \cap A)dt \leq \int_0^\gamma \mu(A)dt + \int_{\gamma}^\infty \mu(\{|f| > t\})dt,
\]
where it is up to you how to choose \( \gamma > 0 \).)
(c) Show by an example that the equivalence in (b) does not hold for \( p = 1 \). Which part fails and why?
(d) Is \( (\text{weak-}L^p, \|\cdot\|_{w,p}) \) a Banach space?

3. (a) Show that \( \lambda_f(\alpha) \) is a decreasing, right continuous function of \( \alpha \).
(b) Show that \( f \in L^p \) if and only if \( \sum_{n=-\infty}^{\infty} 2^{np}\lambda_f(2^n) < \infty \). (Hint: Layer-cake)
(c) (Improved Chebyshev-type estimate) Conclude that if \( f \in L^p \), then \( \lambda_f(\alpha) = o(\alpha^{-p}) \), both as \( \alpha \to 0 \) and \( \alpha \to \infty \).