RV-II Take-Home Midterm

(50 pts) 1. This aim of this problem is to show that for all $1 < p < \infty$, the Fourier series of an $L^p$ function $f$ on $\mathbb{T}$ converges to $f$ in the $L^p$ norm. We had previously ruled out $p = 1$ and $p = \infty$ using the Uniform Boundedness Principle. This time we will show that for any $1 < p < \infty$, $S_N$ is uniformly bounded (in $N$) on $L^p(\mathbb{T})$, which is enough to reach the conclusion.

(a) Define

$$T^\sharp f(x) := f^\sharp(x) := \sum_{n \geq 0} \hat{f}(n)e^{2\pi inx}.$$ 

Show that if $\|T^\sharp\|_{p' \to p} < \infty$ then $\sup_N \|S_N\|_{p' \to p} < \infty$.

(Hint: (i) Write $S_N$ in terms of $T^\sharp$ and multiplication with certain complex exponentials. (ii) What does $f \mapsto e^{2\pi i N x} f$ mean in terms Fourier coefficients?)

(b) Define

$$T^\flat f(x) := f^\flat(x) := -i \sum \text{sgn}(n) \hat{f}(n)e^{2\pi inx}$$

where $\text{sgn}$ is the usual signum function with $\text{sgn}(0) = 0$. By relating $f^\flat$ to $f^\sharp$, show that $T^\sharp$ is bounded if and only if $T^\flat$ is bounded. (By the way, $\sharp$ and $\flat$ have no music-related interpretations here.)

(c) Show that $\|T^\flat\|_{p' \to p'} = \|T^\flat\|_{p' \to p}$, so it suffices to consider $2 \leq p < \infty$ only.

(Hint: show that $(f^\flat, g) = (f, g^\flat)$ and use duality.)

(d) Show that if $f$ is real then $f^\flat$ is also real. Hence argue that it suffices to show that $\|f^\flat\|_p \leq C_p \|f\|_p$ for real-valued trigonometric polynomials $f$.

(e) Show that instead of all $2 \leq p < \infty$ it suffices to consider only $p$ values along a sequence of $p_k \to \infty$.

(f) Consider a real-valued trigonometric polynomial $f$ with zero mean. Show that for any integer $k \geq 1$, we have

$$\int_\mathbb{T} (f + if^\flat)^{2k} dx = 0.$$ 

(g) Fix $k$. Using the binomial theorem, expand out the above expression and consider its real part. Isolate $\int (f^\flat)^{2k}$, and using Hölder’s inequality, show that

$$\|f^\flat\|_{2k}^{2k} \leq \sum_{m=1}^k B_m \|f^\flat\|_{2k-2m}^{2k-2m} \|f\|_{2k}^{2m}$$

for some absolute constants $B_m$, $m = 1, \ldots, k$. 
(h) Let $\alpha = \|f^{}\|_{2k}/\|f\|_{2k}$. Rewrite the above inequality in terms of $\alpha$ and the $B_m$. Show that there is an absolute constant $C_k$ such that $\alpha \leq C_k$, hence conclude that $T^b$ is bounded on $L^{2k}$ and finish the proof of the theorem.

(15 pts) 2. There is a simple rearrangement inequality for oppositely ordered finite sequences. Show that

$$\sum_{k=0}^{n} a^*_{n-k} b^*_k \leq \sum_{k=0}^{n} a_k b_k$$

for all (non-negative) sequences $(a_k)_{k=0}^{n}$ and $(b_k)_{k=0}^{n}$.

(15 pts) 3. Prove the following generalization of the simple rearrangement inequality:

$$\int_X |f_1 f_2 \cdots f_n| d\mu \leq \int_0^\infty f_1^* f_2^* \cdots f_n^* dt.$$ 

(20 pts) 4. Show that the above generalization of the simple rearrangement inequality implies the arithmetic-mean geometric-mean inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

where $a_i \geq 0$.

(Hint keyword: circulant matrix.)