First-passage percolation is a growth model in a random medium introduced by Hammersley and Welsh [10]. We will describe the model on the square lattice $\mathbb{Z}^d$. The random medium consists of nonnegative weights $\{\tau_e\}_{e \in E(\mathbb{Z}^d)}$ attached to the edges of the lattice, where $E(\mathbb{Z}^d) := \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d, |x - y|_1 = 1\}$. The weights can be chosen independently for each edge from a common distribution, or more generally, the medium can be assumed to be stationary and ergodic. A path consists of a sequence of nearest-neighbour vertices, and the total time of a path is the sum of the edge-weights it encounters. The first-passage time $T(x, y)$, is defined to be the infimum over all paths from $x$ to $y$ of the time for each path.

The law of large numbers for $T(x)$ has been the subject of a lot of research over the last 50 years. It involves the existence of the so-called *time-constant* $\mu(x)$ given by

$$\mu(x) := \lim_{n \to \infty} \frac{T([nx])}{n},$$

where $[nx]$ is the closest integer point to $nx$. The limit certainly exists in $d = 1$, since it is simply the usual law of large numbers. One of the most famous and general theorems in $d \geq 2$ is due to Kingman [13]. Motivated by the theory of subadditive processes introduced by Hammersley and Welsh [10] for first-passage percolation, he proved the subadditive ergodic theorem. It follows from this that $T_n(x) \to \mu(x)$ with probability one. However, surprisingly little is known about
the actual value of the time-constant. This has been an open problem for the last several decades. I proved a new formula for the time-constant, and used it to prove several new results about first-passage percolation.

The first-passage time can be viewed as an optimal-control problem on the lattice, and the convergence of $T_n(x)$ to $\mu(x)$ can be seen as a problem of homogenization. In the continuum, such problems are studied in the theory of stochastic homogenization for Hamilton-Jacobi-Bellman (HJB) partial differential equations. This active field of study has seen a lot of progress in the last fifteen years. Some significant results that are relevant to our work include Souganidis [20], Rezakhanlou and Tarver [18], Lions and Souganidis [16], Armstrong et al. [1] and Armstrong and Souganidis [2].

In the first part of my thesis, we consider first-passage percolation in the general stationary-ergodic setting and prove that the time-constant satisfies a HJB equation on $\mathbb{R}^d$. Proving that a continuous, but non-smooth function is a solution of a HJB equation is most easily done using viscosity solution theory [8]. However, this is a continuum theory, and first-passage percolation is on the lattice. We show that there is a smooth optimal-control problem on $\mathbb{R}^d$ that approximates first-passage percolation on the lattice. This allows us to borrow the tools we need from the continuum to prove our theorem.

In the second part of my thesis, we prove a variational formula for the PDE, and then explicitly solve the PDE to obtain the time-constant. The variational formula has several applications: it can be used to compute the time-constant exactly for general edge-weight distributions in some specially generated media; it can be used to prove bounds; and it’s especially well-suited to estimate the difference between time-constants produced by different edge-weight distributions. I’ve presented a selection of my results here.

Before going into the technical details, I’d like to make two acknowledgements. I was introduced to first-passage percolation in the 2nd year of my PhD by Sourav Chatterjee, and worked with him when he was at the Courant Institute. He supported my work even when the ideas were extremely vague and nebulous. I would not have proceeded along this line of inquiry if it weren’t for his encouragement. I received enormous amounts of help from Raghav Varadhan. He found several gaps in my arguments, made uncountably many helpful suggestions, and mentored me from the beginning of my PhD.

2. Background

2.1. First-Passage Percolation. Let us first consider the case of i.i.d edge-weights. Cox and Durrett [6] proved a celebrated result about the relationship between the time-constant and the limit-shape. Let $R_t := \{x \in \mathbb{R}^2 : T([x]) \leq t\}$ be the reachable set. It is a fattened version of the sites reached by the percolation before time $t$. One is interested in the limiting behavior of the set $t^{-1}R_t$ as $t \to \infty$; i.e., one asks if there is a limit-shape. Let $F(t)$ be the cumulative distribution of the edge-weights. Define the distribution $G$ by $(1 - G(t)) = (1 - F(t))^4$. The following theorem holds iff the second moment of $G$ is finite.
Theorem (Cox and Durrett [6]). Fix any \( \epsilon > 0 \). If \( \mu(x) > 0 \) for all \( x \),
\[
\{ x : \mu(x) \leq 1 - \epsilon \} \subset \frac{R_t}{t} \subset \{ x : \mu(x) \leq 1 + \epsilon \} \text{ as } t \to \infty \text{ a.s.} \tag{3}
\]
Otherwise \( \mu \) is identically 0, and for every compact \( K \subset \mathbb{R}^2 \),
\[
K \subset \frac{R_t}{t} \text{ as } t \to \infty \text{ a.s.}
\]

Under the conditions of the above theorem, the set \( B_0 = \{ x : \mu(x) \leq 1 \} \) is known as the limit shape. The extension to \( \mathbb{Z}^d \) was shown by Kesten [12], and Boivin [4] proved the result for stationary-ergodic media. Precise necessary and sufficient conditions are known to determine when \( \mu(x) \) is identically zero [12]. The time-constant can be determined exactly for perhaps two or three different types of edge-weight distributions in a simpler variant of first-passage percolation [11, 19]. There are several beautiful, but pathological special cases studied in detail by Durrett and Liggett [9], Marchand [17], and [3]. However, little else is known in sufficient generality about the time-constant.

It is also worth mentioning that the fluctuations of first-passage percolation have been conjectured to be in the so-called KPZ universality class. This can be proved in a special case of a simpler version called directed first-passage percolation. Here, we restrict attention to the first quadrant of \( \mathbb{Z}^2 \), and consider paths that only go “up-and-right”. Using the asymptotics of orthogonal polynomials, Johansson [11] was able to calculate the time-constant for one very special (but not non-negative) edge-weight distribution. Further, he also showed that fluctuations about the time-constant followed a Tracy-Widom distribution. By the universality hypothesis for such growth models, it’s reasonable to believe that first-passage percolation should have such fluctuations for a much larger class of edge-weight distributions.

2.2. Optimal-Control and Discrete Stochastic Homogenization. As mentioned earlier, it’s useful to view first-passage percolation as an optimal-control problem. Define,
\[
A := \{ \pm e_1, \ldots, \pm e_d \},
\]
where \( e_i \) are the canonical unit basis vectors for the lattice \( \mathbb{Z}^d \). We will write \( \tau(x, \alpha) \) to refer to the weight at \( x \in \mathbb{Z}^d \) along the direction \( \alpha \in A \). The discrete dynamic programming principle (DPP) says that
\[
T(x) = \inf_{\alpha \in A} \{ T(x + \alpha) + \tau(x, \alpha) \}.
\]
We wish to write the DPP as a difference equation in the so-called metric form. Assuming \( \tau(x, \alpha) \) are strictly positive,
\[
\sup_{\alpha \in A} \left\{ -\frac{(T(x + \alpha) - T(x))}{\tau(x, \alpha)} \right\} = 1. \tag{5}
\]
Let’s imagine that we were somehow able to extend \( T(x) \) and \( \tau(x, \alpha) \) as smooth functions on \( \mathbb{R}^d \). Fix \( x \in \mathbb{R}^d \), and Taylor expand (5) at \( \lfloor nx \rfloor \) to get
\[
\sup_{\alpha} \left\{ -\frac{DT([nx] \cdot \alpha + D^2T(\xi)\alpha^2)}{\tau([nx], \alpha)} \right\} = 1.
\]
Substituting $T_n(x) = T([nx])/n$ into (5), we obtain

$$\sup_\alpha \left\{ - \frac{DT_n(x) \cdot \alpha}{\tau([nx], \alpha)} \right\} + O(n^{-1}) = 1.$$  

This equation is the discrete counterpart to the homogenization problem for metric Hamilton-Jacobi-Bellman (HJB) equations in $\mathbb{R}^d$.

For HJB equations on $\mathbb{R}^d$, Lions and Souganidis [16] proved a very general theorem that applies to a large class of Hamiltonians. Their methods rely heavily on the optimal-control interpretation, and we were therefore able borrow several ideas from them. We must also mention the work of Armstrong and Souganidis [2], that focuses specifically on metric Hamiltonians like the one for first-passage percolation. In fact, it was brought to our attention that Armstrong et al. [1] made the following observation: since $T(x,y)$ induces a random metric on the lattice, it’s reasonable to believe that there ought to be some relation to metric HJB equations. This is exactly what we prove.

3. Main Results

3.1. Homogenization Theorem. The results in this section are from the paper Krishnan [14]. Our first result is the homogenization theorem for the time-constant $\mu(x)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let the edge-weights $\tau : \mathbb{Z}^d \times A \times \Omega \to \mathbb{R}$ be

(1) bounded above and below for all $x \in \mathbb{Z}^d$, $\alpha \in A$ and $\omega \in \Omega$; i.e.,

$$0 < a \leq \tau(x, \alpha, \omega) \leq b,$$  

and

(2) stationary-ergodic with respect translation by $\mathbb{Z}^d$.

Theorem 3.1. The time-constant $\mu(x)$ solves a Hamilton-Jacobi equation

$$\bar{H}(D\mu(x)) = 1,$$
$$\mu(0) = 0.$$  

The next result is a variational formula for $\bar{H}(p)$. Let the discrete Hamiltonian for first-passage percolation be

$$\hat{H}(f, p, x, \omega) = \sup_{\alpha \in A} \left\{ \frac{- (f(x + \alpha, \omega) - f(x, \omega)) - p \cdot \alpha}{\tau(x, \alpha, \omega)} \right\}.$$  

Define the set of functions

$$S := \left\{ f : \mathbb{Z}^d \times \Omega \to \mathbb{R} \mid f \text{ has stationary increments, } E[f(x + \alpha) - f(x)] = 0 \forall x \in \mathbb{Z}^d \text{ and } \alpha \in A. \right\},$$  

where the set $A$ is defined in (4). Then,

Theorem 3.2. $\bar{H}(p)$ is a convex, coercive, Lipschitz continuous function given by

$$\bar{H}(p) = \inf_{f \in S} \sup_{\omega \in \Omega} \sup_{x \in \mathbb{Z}^d} \hat{H}(f, p, x, \omega).$$  

(8)
The variational formula tells us that $\varphi(p)$ is positive 1-homogeneous, convex, and $\varphi(p) = 0$ iff $p = 0$. This means that it is a norm on $\mathbb{R}^d$, and indeed, the same is true of $\mu(x)$. By an elementary Hopf-Lax formula for the PDE in Theorem 3.2, Corollary 3.3.

**Corollary 3.3.** $\varphi(p)$ is the dual norm of $\mu(x)$ on $\mathbb{R}^d$, defined as usual by 

$$\varphi(p) = \sup_{\mu(x)=1} p \cdot x.$$ 

3.2. **Applications of the Variational Formula.** The results in this section are from the (forthcoming) paper Krishnan [15].

The dual problem, and a scheme to produce bounds. The key to the homogenization theorem is the dual variational problem that comes from the so-called cell-problem in the theory of homogenization. We can use this dual problem quite effectively to produce bounds for the time-constant. Before we can state the results, we need to define the basic notation used in optimal-control theory. Let the state $y_{\alpha,x}(k)$ of a control system evolve in discrete time via the difference equation

$$y_{\alpha,x}(j + 1) = y_{\alpha,x}(j) + \alpha(j) \quad \forall j \geq 0,$$

$$y_{\alpha,x}(0) = x,$$  

where the controls $\alpha$ lie in the set

$$\mathcal{A} := \{\alpha : \mathbb{Z}^+ \to A\}.$$

Suppose we have edge-weights $\tau(x,\alpha)$ as in first-passage percolation, and in addition, a set of running costs $l(x,\alpha) : \mathbb{Z}^d \times A \to \mathbb{R}$. Assume that $\tau(x,\alpha)$ is positive and bounded as in (6) and that $|l(x,\alpha)| \leq C$. Let $u_0 : \mathbb{Z}^d \to \mathbb{R}$ be any bounded function. The discrete finite time-horizon optimal-control problem can be described as follows. The total cost of a path is the sum of the running costs it encounters on the edges, plus a final cost $u_0$ depending on its end-point. Taking an infimum of the total cost over all paths starting at $x$ and taking time less than $t$, we get the function $u(x,t)$. We give a precise definition for $u(x,t)$ below.

For a control $\alpha \in \mathcal{A}$, let

$$\mathcal{T}_{x,k}(\alpha) = \sum_{i=1}^{k} \tau(y_{\alpha,x}(i),\alpha(i))$$

be the total time of the path $y_{\alpha,x}$ for $k$ steps. Similarly, define the total cost to be

$$\mathcal{I}_{x,k}(\alpha) = \sum_{i=0}^{k} l(y_{\alpha,x}(i),\alpha(i)).$$

Then,

$$u(x,t) = \inf_{\alpha \in \mathcal{A}} \inf_{k \in \mathbb{Z}^+} \left\{ \mathcal{T}_{x,k}(\alpha) + u_0(y_{\alpha,x}(k)) : \mathcal{T}_{x,k}(\alpha) \leq t \right\}. \quad (12)$$

When $l(x,\alpha) = p \cdot \alpha$, we have

**Proposition 3.4.** for any $R > 0$

$$\lim_{t \to \infty} \frac{u(x,t)}{t} = -\varphi(p) \quad a.s.,$$

where $\varphi(p)$ is the dual of the time-constant in Corollary 3.3.
Prop. 3.4 is well-known in the continuum. As far as we know, this is a new observation for discrete first-passage percolation.

Prop. 3.4 can be used to produce bounds. We divide the medium into finite boxes, and consider a fixed strategy in each box. When the path exits one box and enters another, the strategy “renews” itself. Since the total cost and time spent in each box is independent of other boxes, we can use the renewal theorem to explicitly compute the asymptotic cost of the path.

More precisely, consider the set of vertices \( B_n := \{ x \in \mathbb{Z}^d : |x|_\infty \leq n \} \), and the set of edges \( E_n := \{ (x,y) \in E(\mathbb{Z}^d) : x \in B_{n-1}, y \in B_n \} \). We’ll only consider paths that start at 0, use edges in \( E_n \), and exit at the boundary \( \partial B_n := B_n \setminus B_{n-1} \).

Let’s suppose that we have a strategy that depends only on the edge-weights in \( E_n \), and gives as output the point of exit on the boundary \( \partial B_n \). Denote this strategy function by \( s(E_n, \omega) \). Let \( t_s = T(0, s(E_n, \omega)) \) be the time of exit from the box \( B_n \). Due to the special structure of \( l(x, \alpha) \), the total cost of the path is \( I = p \cdot s(E_n, \omega) \).

Once we exit the box \( B_n \), the strategy renews itself by translating the origin to the point of exit. Then, the renewal theorem gives

**Proposition 3.5.**

\[
\mathbb{H}(p) \geq \sup_s p \cdot E[s(E_n, \omega)] / E[t_s].
\]

By duality, this gives us an upper bound for the time-constant. For any fixed strategy, the expectations can be computed explicitly. We present some explicit examples in Krishnan [15].

**Comparison Results.** The variational formula is particularly well-suited to prove comparison theorems. Again, let us look at the special case where the edge-weights come from one single distribution, and are chosen independently. Let \( F_1 \) and \( F_2 \) be two distributions on \([0, \infty)\) with finite mean. One would like to be able compare the corresponding time-constants \( \mu_1 \) and \( \mu_2 \).

One of the best-known comparison results is due to van den Berg and Kesten [21]. We need the following definition before we can state the theorem.

**Definition 3.6 (Stochastic Domination).** If \( F_1(x) \leq F_2(x) \) \( \forall x \), we say that \( F_2 \) dominates \( F_1 \). If, \( F_1 \neq F_2 \) identically, we say that \( F_2 \) strictly dominates \( F_1 \).

If \( F_2 \) dominates \( F_1 \), it’s very easy to prove that \( \mu_1(x) \leq \mu_2(x) \). However, it’s not clear if strict inequality holds.

**Theorem ([21]).** Suppose \( F_1 \) and \( F_2 \) do not have a large atom on the minimum of their support, and suppose \( F_2 \) strictly dominates \( F_1 \). Then,

\[
\mu_1(e_1) < \mu_2(e_1).
\]

We will not quantify “large” here, and refer to van den Berg and Kesten [21] for the details. It is worth remarking here that strict inequality in Prop. 3.4 is related to the existence of what is usually called a corrector in the theory of stochastic homogenization. The corrector is a function that’s of great importance in the theory of stochastic homogenization. In our setting, we believe that correctors exist; see Theorem 3.8.
Such comparison inequalities are all that was known until our work. We computed a quantitative estimate of the distance between $\mu_1$ and $\mu_2$ in terms of the distance between the two distributions $F_1$ and $F_2$. We state the easy-to-prove version of the theorem that assumes that the distributions have nicely behaved densities supported on a compact set. Let $0 < a_i = \min(\text{supp}(F_i))$ and $b_i = \max(\text{supp}(F_i)) < \infty$, $i = 1, 2$, where sup denotes the support of the distribution. Let each $F_i$ have density $\rho_i$, and let $\rho^* = \min \min_{a_i \leq x \leq b_i} \rho_i(x)$. Assume that $\rho^* > 0$. Let $d_{Kol}(F_1, F_2) = \sup_x |F_1(x) - F_2(x)|$

be the Kolmogorov-Smirnov distance between the two distributions.

**Theorem 3.7.** For all $x \in \mathbb{R}^d$, 

$$|\mu_1(x) - \mu_2(x)| \leq 2 \frac{b_1 b_2}{a_1 a_2} d_{Kol}(F_1, F_2) \rho^* |x|_1.$$ 

This gives a rate of convergence for the weak continuity theorem of Cox and Kesten [7].

**Algorithm to Calculate the Time-Constant in Some Special Cases.** Next, we show how to explicitly calculate the limit shape when a special symmetry is present. Let $\{V_{e_1}, \ldots, V_{e_d}\}$ be measure preserving ergodic transformations on $\Omega$. These generate the random medium in the following way. Let $t(\alpha, \omega)$ be a random vector distributed according to the marginal edge-weight distribution at the origin. For example, it could consist of $d$ i.i.d. edge-weights, one for each direction. Let $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. Then, 

$$\tau(x, \alpha, \omega) = t(\alpha, V_{e_1}^{x_1} \cdots V_{e_d}^{x_d}, \omega).$$

Let us assume for simplicity that $V_1 = \cdots = V_d = V$. This means that for each $\omega$ and $\alpha$, the function $\tau(\cdot, \alpha, \omega)$ is constant along the hyperplanes $\{x \in \mathbb{Z}^d : \sum_{i=1}^d x_i = z\}$ for each $z \in \mathbb{Z}$. Despite this symmetry, the medium is still quite random, and it’s not so obvious what the time-constant is. However, the variational formula is tremendously simplified. With $A_+ = \{e_1, \ldots, e_d\}$, we redefine the discrete Hamiltonian for $t \in \mathbb{R}$, $p \in \mathbb{R}^d$ to be 

$$\hat{H}(t, p, \omega) = \sup_{\alpha \in A_+} \frac{|t + p \cdot \alpha|}{\tau(0, \alpha, \omega)}. \quad (13)$$

In the following, we will write $\hat{H}(f, \omega)$ and drop reference to $p$ since it’s irrelevant to our arguments. Define the set of functions $F := \{f : \Omega \to \mathbb{R}, E[f] = 0\}$. Then, the variational formula in Theorem 3.2 becomes

$$\overline{H}(p) = \inf_{f \in F} \text{ess sup} \hat{H}(f(\omega), p, \omega). \quad (14)$$

$\hat{H}(\cdot, \omega)$ is a convex function for each $\omega$, and moreover, it has a unique minimum. The following algorithm produces a minimizer for the variational problem. We present the algorithm in its entirety to convince the reader that it’s completely explicit and constructive. Our suggestion is to skip it the first time around and go straight to Theorem 3.8.
(1) Start with any $f_0 \in F$, for example, $f_0 = 0$. Let $\mu_0 = E[\hat{H}(f_0, \omega)]$, and let
\[ d = \text{ess sup}_{\omega \in \Omega} \hat{H}(f_0, \omega) - \mu_0. \]
If $d = 0$, stop.

(2) Define the sets
\[ M1N_0 = \{ \omega : \hat{H}(f_0, \omega) = \min_x \hat{H}(x, \omega) \}, \]
\[ S = \{ \omega : \hat{H}(f_0, \omega) > \mu_0 \}, \]
\[ I = \{ \omega : \hat{H}(f_0, \omega) < \mu_0 \}. \]
If
\[ \text{ess sup}_{\omega \in M1N_0} \hat{H}(f_0, \omega) = \text{ess sup}_{\omega \in \Omega} \hat{H}(f_0, \Omega), \]
stop.

(3) Let $\Delta f^*(\omega)$ be such that
\[ \hat{H}(f_0 + \Delta f^*(\omega), \omega) = \min_x \hat{H}(x, \omega). \]
Define the sets
\[ S_+ = \{ \omega \in S : D\hat{H}(f_0, \omega) \subset (-\infty, 0) \}, \]
\[ S_- = \{ \omega \in S : D\hat{H}(f_0, \omega) \subset (0, \infty) \}, \]
where $D\hat{H}$ represents the subderivative set of the (convex) function. Let
\[ \Delta f(\omega) = \begin{cases} 
-\alpha(\hat{H}(f_0, \omega) - \mu_0) \lor \Delta f^*(\omega) & \omega \in S_+ \\
\alpha(\hat{H}(f_0, \omega) - \mu_0) \land \Delta f^*(\omega) & \omega \in S_- \\
\alpha(\hat{H}(f_0, \omega) - \mu_0) & \omega \in I \\
0 & \text{elsewhere}
\end{cases}, \]
where
\[ \xi = \frac{\int_{S_+ \cup S_-} \Delta f(\omega) \mathbb{P}(d\omega)}{\int \Delta f(\omega) \mathbb{P}(d\omega)}. \]
Let $f_1 = f_0 + \Delta f(\omega)$.

(4) If
\[ \text{ess sup}_{\omega \in \Omega} \hat{H}(f_1, \omega) > \text{ess sup}_{\omega \in \Omega} \hat{H}(f_0, \omega) - \frac{da}{2b}, \]
stop. If not, go back to the first step with $f_0 = f_1$ and repeat.

**Theorem 3.8.** There are three possibilities for the algorithm:

(1) If it terminates in a finite number of steps with $d = 0$, we have a minimizer that’s a corrector.
(2) If it terminates in a finite number of steps with $d > 0$, we have a minimizer that’s not a corrector.
(3) If it does not terminate, we produce a corrector in the limit.
The algorithm has an important implication for numerical computations. Suppose the vector \( \vec{\tau}(\omega) := (\tau(0,e_1,\omega), \ldots, \tau(0,e_d,\omega)) \) takes at most \( n \) different values on \( \Omega \). Call these \( n \) vectors \( v_1, \ldots, v_n \). Let

\[
A_i = \{ \omega \in \Omega : \vec{\tau}(\omega) = v_i \},
\]

be a partition of the probability space. Then, we can restrict the set \( F \) of functions to be

\[
F := \left\{ (f_1, \ldots, f_2) \in \mathbb{R}^n : \sum_i f_i \mathbb{P}(A_i) = 0 \right\},
\]

and the algorithm continues to produce a minimizer. But this means that

\[
\overline{H}(p) = \min_{f \in F} \max \{ \hat{H}(f_1), \ldots, \hat{H}(f_n) \},
\]

is a finite dimensional constrained convex optimization problem in \( \mathbb{R}^n \). This is as explicit as it gets. We can solve this by our favorite numerical method instead of the rather inefficient abstract algorithm.

4. Future Work

More general iterative schemes. If we remove the symmetry assumption, the medium is truly random. It’s a challenging problem to come up an explicit algorithm to produce a minimizer. Can the algorithms be tuned to produce correctors under the appropriate assumptions on the edge-weights?

Regularity (and strict convexity). We present some exact limit shapes computed using the algorithm. Consider the following cases:

1. \( \tau(0,\alpha) \in \{a,b\} \), medium is periodic.
2. \( \tau(0,\alpha) \in \{a,b\} \), medium is i.i.d.
3. \( \tau(0,\alpha) \in \{a,c,b\} \), where \( a < c < b \), and the medium is i.i.d.

The cases are chosen to be progressively more random (whatever that means). The corresponding limiting Hamiltonians are shown in Fig. 1. We observe two effects of increasing randomness. First, the speed of percolation increases since there are more opportunities to sneak through the medium. Second, the Hamiltonian become smoother in the (1,1) direction — this is the most random direction because of the symmetry of the medium. If the Hamiltonian \( \overline{H}(p) \) is \( C^1 \), then its dual, the time-constant, is strictly convex. The limit-shape has been conjectured to be strictly convex if the edge-weights are “nice enough” —this has been an open problem for quite a while.

In summary, we have a classical problem in the calculus of variations. There is a variational representation for the limit Hamiltonian, and under appropriate assumptions on the edge-weights, we’d like to prove that \( \overline{H}(p) \) is \( C^1 \). Since we have an explicit iteration that produces the minimizer, we ought to be able to prove regularity for the minimizer, at least under special conditions. This will prove regularity for \( \overline{H}(p) \).
Figure 1. Level sets of $H(p) = 1$ for all three cases of edge-weights.

Quasi-periodic medium. One of the main difficulties in working with the variational formula is that the set $S$ of functions in the variational formula (Theorem 3.2) is hard to describe. However, if the probability space is $S^1$, and the translation group is generated by irrational rotations, the ergodic system is quasi-periodic. In this case, we hope to use Kolmogorov-Arnold-Moser theory (or more precisely, what’s known as the stochastic weak-KAM theory) to analyze the problem.

Optimal strategies in finite boxes. The method to produce bounds in Prop. 3.5 suggests the following question: what’s the strategy that attains the supremum? If we take the box size to infinity, do we obtain the limit Hamiltonian $H(p)$? Since we’re now optimizing on the level of expectations, there is hope to solve this problem.

Fluctuations in first-passage percolation. As stated earlier, the fluctuations of the time-constant are thought to be in the KPZ universality class. Rigorous results for fluctuations are only known for very special cases of edge-weights. Such integrable systems, so-to-speak, are of great interest in the community [11, 5]. This is an area I’m starting to explore.

References


