Homework 4

Due Monday, July 13th at the beginning of class

1. [Taken from DeGroot] In a certain city 30 percent of the people are Conservative, 50 percent are Liberals, and 20 percent are Independent. We also know that 65 percent of Conservatives voted, 82 percent of Liberals voted, and 50 percent of Independents voted. If a person in the city is selected at random and we learn they didn’t vote, what is the probability they are Liberal?

*Solution.* Let $C, L, I$ be the events denoting Conservative, Liberal, and Independent, respectively. Let $V$ be the event denoting voting. Then we have

\[
P(L|V^c) = \frac{P(V^c|L)P(L)}{P(V^c|L)P(L) + P(V^c|I)P(I) + P(V^c|C)P(C)}
\]

\[
= \frac{.18 \cdot .5}{.18 \cdot .5 + .5 \cdot .2 + .35 \cdot .3}
\]

\[
\approx .305.
\]

2. You roll a fair $n$-sided die $k$ times.

   (a) Give a sample space and an associated probability measure.

   (b) What is the chance that your rolls are strictly increasing? That is, the second roll is larger than the first, and third is larger than the second, etc.

*Solution.*

   (a) $S = \{(d_1, \ldots, d_k) : 1 \leq d_i \leq n\}$ where all outcomes are equally likely.

   (b) $\binom{n}{k}/n^k$.

3. Give a counting argument to show that

\[
\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}
\]

where $2 \leq k \leq n$.

*Solution.* The number of subsets of size 2 taken from a set of size $n$ is given by $\binom{n}{2}$. Suppose the elements are broken into two groups, the first of size $k$ and the remainder of size $n-k$. Then to choose 2 elements we could either pick two from the first group, two from the second group, or one from each group. These three options have $\binom{k}{2}$, $k(n-k)$ and $\binom{n-k}{2}$ choices each, respectively.
4. For any event $A$, let an $A$-certificate be a contract that requires the seller to give the buyer $1000$ if the event $A$ occurs, and $0$ otherwise. For a particular (sports-based) experiment, your friend is willing to buy or sell any $A$-certificate for $P_{\text{friend}}(A) \cdot 1000$ dollars, where $P_{\text{friend}}$ reflects your friend’s belief system. Assume there are disjoint events $B, C$ such that

$$P_{\text{friend}}(B) + P_{\text{friend}}(C) \neq P_{\text{friend}}(B \cup C).$$

Show how making a few trades with your friend can earn you some money. This shows how a belief system that doesn’t adhere to the axioms of probability can be exploited in the context of betting.

Solution. Let $B, C$ be the disjoint events described in the statement. If

$$P_{\text{friend}}(B) + P_{\text{friend}}(C) > P_{\text{friend}}(B \cup C)$$

then we should sell him a $B$-certificate and a $C$-certificate and purchase a $(B \cup C)$-certificate from him. Due to the inequality above, this set of transactions nets us

$$1000(P_{\text{friend}}(B) + P_{\text{friend}}(C) - P_{\text{friend}}(B \cup C))$$

dollars. Analogously, if

$$P_{\text{friend}}(B) + P_{\text{friend}}(C) < P_{\text{friend}}(B \cup C)$$

then we sell a $(B \cup C)$-certificate and purchase a $B$-certificate and a $C$-certificate again netting us money.

If $(B \cup C)^c$ occurs, none of the contracts pay and we keep our money. If $(B \cup C)$ occurs then he must pay us $1000$ and we must pay him $1000$ so again we make money in total. Stated a different way, holding a $B$-certificate and a $C$-certificate is equivalent to holding a $(B \cup C)$-certificate, so they should have the same price.

5. Suppose you roll a 100-sided die $n$ times. The die isn’t fair, and gives the value $i$ with probability $p_i$ where $\sum_{i=1}^{100} p_i = 1$. What is the probability your $n$th roll is different from all previous rolls? [Hint: Condition on the value of the last roll and apply the LOTP.]

Solution. Let $A_i$ denote the event that the last roll is $i$, and let $B$ denote the event that the last roll is different from all previous. Then we have

$$P(B) = \sum_{i=1}^{100} P(B|A_i)P(A_i).$$

Note that

$$P(A_i) = p_i, \quad P(B|A_i) = \frac{P(B \cap A_i)}{P(A_i)} = \frac{(1 - p_i)^{n-1}p_i}{p_i} = (1 - p_i)^{n-1}$$
giving

\[ P(B) = \sum_{i=1}^{100} (1 - p_i)^{n-1} p_i. \]

6. Suppose an urn has \( M \) black balls and \( N \) white balls. You draw \( k \) balls from it without replacement where \( 1 \leq k \leq M + N \).

(a) Give a sample space and a probability measure. [Treat every ball as distinct, and either use subsets or ordered sequences.]

(b) What is the probability that \( j \) of the \( k \) balls are black?

**Solution.**

(a) Suppose the balls are numbered 1, \ldots, \( M + N \) where the first \( M \) are black. If we use unordered sets as our sample space then we get

\[ S = \{ B : B \subset \{1, \ldots, M + N\}, |B| = k \} \]

with all outcomes equally likely. If we use ordered sequences then we get

\[ S = \{ (b_1, \ldots, b_k) : b_i \in \{1, \ldots, M + N\}, b_i \text{ are distinct} \} \]

where again all outcomes are equally likely.

(b) For the unordered sample space we get

\[ \frac{\binom{M}{j} \binom{N}{k-j}}{\binom{M+N}{k}}. \]

For the ordered sample space we get

\[ \frac{\binom{M}{j} \binom{N}{k-j} k!}{(M + N)(M + N - 1) \cdots (M + N - k + 1)} \]

which is the same answer.