Exercise 1 - based on S95#1

A linear transformation $T$ from $\mathbb{R}^3 \to \mathbb{R}^3$ has $Tv_i = w_i$ for $i = 1, 2, 3$, where

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, & v_2 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, & v_3 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\
w_1 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, & w_2 &= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, & w_3 &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},
\end{align*}
\]

(a) Find the matrix that maps a vector in $\mathbb{R}^3$ expressed in standard coordinates to a vector containing the coordinates w.r.t $v_i$.

(b) Find the matrix $B$ that represents $T$ in the standard basis.

(c) Find the matrix $A$ that represents $T$ in the basis $v_1, v_2, v_3$

Exercise 2 - based on J00#2

Let $n \geq 2$. Set $V_n$ equal to the vector space of functions $p(x)e^{3x}$ where $p(x)$ is a polynomial of degree at most $n$. Define the linear map $L : V_n \to V_n$ by

\[
L(f) = \frac{d^2 f}{dx^2} - 6 \frac{df}{dx} + 9f.
\]

(a) Give bases for $\ker(L)$ and $\text{im}(L)$.

(b) Write the matrix of $L$ with respect to the basis \{e^{3x}, xe^{3x}, \ldots, x^ne^{3x}\}.

(c) What are the eigenvalues of $L$?

(d) For $n = 3$, express $L$ in Jordan form.
Exercise 3 - Nilpotent operators: based on J03#3

A nilpotent operator $T$ satisfies the condition $T^q = 0$ for some positive $q \in \mathbb{Z}$.

(a) Prove that any square upper triangular matrix with diagonal elements zero is nilpotent

(b) Conversely, prove that if a nilpotent $T$ is defined on a finite dimensional vector space $V$ then there exists a basis for $V$ such that the matrix representation of $T$ is upper triangular with diagonal elements zero

(c) Calculate the characteristic polynomial and the spectrum of $T$ under the conditions in (b).

Exercise 4 - S04#2

Let $A_1 : \mathbb{R}^{3^1} \to \mathbb{R}^{1^4}$, $A_2 : \mathbb{R}^{3^{1^1}} \to \mathbb{R}^{1^0}$, $A_3 : \mathbb{R}^{3^{1^1}} \to \mathbb{R}^4$ be three linear maps.

(a) Show that $\text{Ker}(A_1) \cap \text{Ker}(A_2) \cap \text{Ker}(A_3) \neq \emptyset$.

(b) Suppose $A : V \to V$ is a linear map with $A^5 = 0$ and $\dim \text{Ker}A = 7$. What is the largest possible value for $\dim V$?

Exercise 5 - Polar decomposition (cf. Evans-Gariepy) and Singular Value Decomposition (SVD)

Let $M \in \mathbb{R}^{m \times n}$ be a matrix. In this exercise we will show that

(i) If $n \leq m$, there exists a symmetric $n \times n$ matrix $S$ and an orthogonal $n \times m$ matrix $O$ such that

$$M = OS.$$

(ii) If $n \geq m$, there exists a symmetric $m \times m$ matrix $S$ and an orthogonal $n \times m$ matrix $O$, such that

$$M = S O^T.$$

(a) First, suppose $n \leq m$. Show that $C = M^* M$ is symmetric, nonnegative definite.

(b) So, $C$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_n \geq 0$, and an orthogonal basis of eigenvectors $x_k$, $k = 1, \ldots, n$. Define $\lambda_k = \sqrt{\mu_k}$. Prove that there is an orthonormal set $\{z_k\}_{k=1}^n$ in $\mathbb{R}^m$ such that

$$L x_k = \lambda_k z_k$$
(c) Construct $O$ and $S$ of part (i).

(d) How does (ii) follow from (i)?

(e) How does the decomposition relate to the Singular Value Decomposition $M = U\Sigma V^T$, with $U$ an orthogonal $m \times m$ matrix, $\Sigma$ an $m \times n$ matrix, of which all diagonal entries are nonnegative, and the off-diagonal entries all vanish, and $V$ is an orthogonal $n \times n$ matrix.

**Exercise 6 - Actually perform the computation for the polar decomposition: S04 # 3**

Write $C$ as a product $C = PO$ where $O$ is an orthogonal matrix and $P$ is positive definite symmetric

$$C = \begin{pmatrix} 4 & 2 & 0 \\ 5 & -1 & \sqrt{2} \\ 1 & 1 & 0 \end{pmatrix}$$

**Exercise 7 - Prove the rank-nullity theorem by picking bases**

Prove that for a linear map $L : V \rightarrow W$,

$$\dim \ker L + \dim \text{Im} L = \dim V$$