Write up and turn in solutions to the highlighted problems from the list below. However, solve all other problems as well. The quizzes will cover the complete list of homework problems.

Due date: Friday, October 18, in the beginning of the recitation. NO LATE HOMEWORK WILL BE ACCEPTED.

Problems.

1. Let $G$ be a group and let $Z(G) = \{ z \in G : zx = xz \text{ for all } x \in G \}$ be the center of $G$ (we encountered the center of $G$ is problem 15 on page 47 of Herstein’s book, which was part of homework 4. We showed there that $Z(G) \leq G$).
   
   (a) Let $H \leq Z(G)$ be a subgroup of $Z(G)$. Show that $H$ is normal in $G$.
   
   (b) Assume that $H \leq Z(G)$ and prove that if $G/H$ is a cyclic group then $G$ is Abelian.
       
       Hint: Let $Hx$ be a generator of $G/H$. What is $\langle H \cup \{x\} \rangle$?

2. Let $G$ be a group and $g, h \in G$. Define the commutator of $g$ and $h$ as $[g, h] = ghg^{-1}h^{-1}$. Then define the commutator subgroup, denoted $[G, G]$, of $G$ as the subgroup generated by all commutators of elements of $G$, i.e. $[G, G] = \langle \{[g, h] : g, h \in G\} \rangle$.
   
   (a) What does an element of $[G, G]$ look like?
       
       Hint: It is not enough to only consider elements of the form $[g, h]$.
   
   (b) Prove that $[G, G]$ is a normal subgroup of $G$.

   (c) Prove that $G/[G, G]$ is an Abelian group.

   (d) Let $N \triangleleft G$ and assume that $G/N$ is a Abelian. Prove that $N \supseteq [G, G]$.

   (e) Assume that $H \leq G$ and $H \supseteq [G, G]$. Deduce that $H \triangleleft G$.

3. Let $\mathbb{H}$ be the $3 \times 3$ Heisenberg group over the real numbers, i.e.
   
   $$ \mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}. $$

   We have seen in class that $\mathbb{H}$, equipped with the usual matrix product, is a group. Compute the center of $\mathbb{H}$, i.e. the subgroup $Z(\mathbb{H}) \triangleleft \mathbb{H}$, and show that it is isomorphic to the additive real numbers ($\mathbb{R}, +$). Show that $\mathbb{H}/Z(\mathbb{H})$ is isomorphic to the additive complex numbers ($\mathbb{C}, +$).


   Hint for problem 17 on page 66 of Herstein: consider the mapping $f : \mathbb{R} \to \mathbb{C}$ given by $f(x) = e^{2\pi ix}$. 