RIGOROUS PROOF THAT \((e^x)' = e^x\)

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The number \(e \in \mathbb{R}\) was defined in class as follows:

\[
e \overset{\text{def}}{=} \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.
\]  

(1)

In particular, we proved rigorously in class that the limit in (1) exists.

Having defined the number \(e\), for \(x \in \mathbb{R}\) the number \(e^x\) was defined rigorously using the least upper bound axiom in Homework 2. In particular, it was shown that for all \(x, y \in \mathbb{R}\) we have \(e^{x+y} = e^x e^y\), and a similar (even simpler) argument shows that \(e^{xy} = (e^x)^y\).

**Lemma 0.1.** Assume that \(\{a_n\}_{n=1}^\infty \subseteq (0, \infty)\) and \(\lim_{k \to \infty} a_n = e\). Then for all \(x \in \mathbb{R}\) we have \(\lim_{n \to \infty} a_n^x = e^x\).

**Proof.** It suffices to deal with the case \(x > 0\), since the identity \(a_n^x = 1/a_n^{-x}\) will then imply the required result for negative \(x\). Fix \(\varepsilon > 0\). Choose \(m \in \mathbb{N}\) such that \(m > x\) and define \(\delta = \min\{1, \varepsilon/2^m\}\). Since we are assuming that \(\lim_{n \to \infty} a_n/e = 1\), there exists \(N \in \mathbb{N}\) such that for all \(n \in \mathbb{N}\) satisfying \(n \geq N\) we have

\[
e(1 - \varepsilon) < a_n < e(1 + \varepsilon).
\]

So,

\[
(1 - \delta)^m \leq (1 - \delta)^x < \frac{a_n^x}{e^x} < (1 + \delta)^x \leq (1 + \delta)^m.
\]

By Bernoulli’s inequality,

\[
(1 - \delta)^m \geq 1 - m\delta \geq 1 - 2^m\delta \geq 1 - \varepsilon,
\]

and by the binomial formula,

\[
(1 + \delta)^m = 1 + \binom{m}{1}\delta + \binom{m}{2}\delta^2 + \cdots + \binom{m}{m}\delta^m
\]

\[
< 1 + \delta \left(1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m}\right) = 1 + \delta(1 + 1)^m = 1 + \delta 2^m \leq 1 + \varepsilon.
\]

Hence,

\[
\left|\frac{a_n^x}{e^x} - 1\right| < \varepsilon,
\]

and we have therefore proved that \(\lim_{n \to \infty} a_n^x/e^x = 1\), as required. \(\square\)

**Lemma 0.2.** Assume that \(\{a_k\}_{k=1}^\infty \subseteq (0, \infty)\) and \(\lim_{k \to \infty} a_k = \infty\). Then

\[
\lim_{k \to \infty} \left(1 + \frac{1}{a_k}\right)^{a_k} = e.
\]
Proof. Define three sequences \( \{ x_n \}_{n=1}^{\infty}, \{ y_n \}_{n=1}^{\infty}, \{ z_n \}_{n=1}^{\infty} \subseteq (0, \infty) \) as follows:

\[
x_n \overset{\text{def}}{=} \left( 1 + \frac{1}{n} \right)^n, \quad y_n \overset{\text{def}}{=} \left( 1 + \frac{1}{n+1} \right)^n, \quad z_n \overset{\text{def}}{=} \left( 1 + \frac{1}{n} \right)^{n+1}.
\]

Then using \( \square \) we see that

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} \frac{x_{n+1}}{1 + \frac{1}{n+1}} = e = e,
\]

and

\[
\lim_{n \to \infty} z_n = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) x_n = 1 \cdot e = e.
\]

Now, let \( n_k \in \mathbb{N} \) be the largest integer such that \( n_k \leq a_k \). Then for all \( k \) we have \( n_k \leq a_k < n_k + 1 \). This implies that \( \lim_{k \to \infty} n_k = \infty \) and

\[
1 + \frac{1}{n_k + 1} < 1 + \frac{1}{a_k} \leq 1 + \frac{1}{n_k}.
\]

Hence,

\[
y_{n_k} = \left( 1 + \frac{1}{n_k + 1} \right)^{n_k} \leq \left( 1 + \frac{1}{a_k} \right)^{n_k} \leq \left( 1 + \frac{1}{a_k} \right)^{a_k} \leq \left( 1 + \frac{1}{n_k} \right)^{n_k+1} = z_{n_k}.
\]

Since \( \lim_{k \to \infty} y_{n_k} = \lim_{k \to \infty} z_{n_k} = e \), the required result follows from an application of the squeeze theorem.

\[ \square \]

**Corollary 0.3.** For every \( x \in \mathbb{R} \) we have

\[
\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x.
\]

Proof. Assume first that \( x > 0 \). Then \( \lim_{n \to \infty} \frac{n}{x} = \infty \), and therefore Lemma 0.2 implies that if we denote \( b_n = (1 + x/n)^{n/x} \) then \( \lim_{n \to \infty} b_n = e \). An application of Lemma 0.1 shows that

\[
\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = \lim_{n \to \infty} b_n^n = e^x.
\]

Assume now that \( x < 0 \) and denote \( y = -x > 0 \). By what we have just proved we know that \( \lim_{n \to \infty} (1 + y/n)^n = e^y \). For \( n > y \) we have \( y/n < 1 \), so me may use Bernoulli’s inequality to deduce that

\[
1 \geq \left( 1 - \frac{y^2}{n^2} \right)^n \geq 1 - \frac{y^2}{n}.
\]

By the squeeze theorem this implies that

\[
\lim_{n \to \infty} \left( 1 - \frac{y^2}{n^2} \right)^n = 1.
\]

Hence,

\[
\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = \lim_{n \to \infty} \left( 1 - \frac{y}{n} \right)^n = \lim_{n \to \infty} \frac{(1 - y/n)^n (1 + y/n)^n}{(1 + y/n)^n} = \lim_{n \to \infty} \frac{(1 - y^2/n^2)^n}{(1 + y/n)^n} = \frac{1}{e^y} = e^x,
\]

as required.
Lemma 0.4. For all \( x \in \mathbb{R} \) we have \( e^x \geq 1 + x \), \( e^x \leq \frac{1}{1-x} \).

Proof. For \( n > |x| \) we have \( x/n > -1 \). We can therefore apply Bernoulli’s inequality as follows:
\[
\left(1 + \frac{x}{n}\right)^n \geq 1 + n \frac{x}{n} = 1 + x.
\]
In conjunction with Corollary 0.3 this implies (2).

Assume now that \( x < 1 \), and use what we have just proved with \( x \) replaced by \(-x\). We get \( e^{-x} \geq 1 - x \), and since \( 1 - x > 0 \), this implies (3). \( \square \)

Theorem 0.5. The function \( f(x) = e^x \) is differentiable on \( \mathbb{R} \) and \( f'(x) = e^x \) for all \( x \in \mathbb{R} \).

Proof. For every \( x, h \in \mathbb{R} \) we have
\[
\frac{f(x + h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = e^x \frac{e^h - 1}{h}.
\]
By Lemma 0.4 for all \( h \in (0, 1) \) we have
\[
1 \leq e^h - 1 \leq \frac{1}{1-h} - 1 = \frac{1}{1-h},
\]
and for \( h \in (-1, 0) \) we have
\[
\frac{1}{1-h} = \frac{1}{1-h} - 1 \leq \frac{e^h - 1}{h} \leq 1.
\]
By the squeeze theorem for functions it follows that
\[
\lim_{h \to 0} \frac{e^h - 1}{h} = 1.
\]
Hence, by (4) we have
\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = e^x.
\] \( \square \)