Rationally trivial principal homogeneous spaces, purity and arithmetic of reductive group schemes over extensions of two-dimensional local regular rings

Yevesy Nisnevich

Abstract — Let $X$ be a regular scheme, $G$ a reductive group scheme over $X$. Serre and Grothendieck conjectured that any rationally trivial $G$-torsor is locally trivial in the Zariski topology of $X$. We prove this conjecture when dim $X=2$ and $G$ is quasi-split over $X$.

Espaces homogènes principaux rationnellement triviaux, pureté et arithmétique des schémas en groupes réductifs sur les extensions d’anneaux locaux réguliers de dimension 2

Résumé — Soit $X$ un schéma régulier, et soit $G$ un schéma en groupes réductif sur $X$. Serre et Grothendieck ont conjecturé que tout $G$-torsor rationnellement trivial est localement trivial pour la topologie de Zariski de $X$. Nous démontrons cette conjecture dans le cas où $X$ est de dimension 2 et $G$ quasi-déployé.

Version française abrégée — Cette Note fait suite à [9], dont on garde la terminologie et les notations. Soient $X$ un schéma noethérien intègre et régulier, $K=R(X)$ le corps des fonctions rationnelles sur $X$ et $G$ un schéma en groupes réductif sur $X$. On a la suite de cohomologie de $G$:

$$1 \rightarrow H^1(X_{\text{zar}}, G) \rightarrow H^1(X_{\text{et}}, G) \rightarrow H^1(K, G).$$

Conjecture 1.2 (Serre [12], Grothendieck [12], [5]). — La suite (1.1) est exacte.

Lorsque $X$ est de dimension 1, ou local hensélien, et $G$ est un $X$-groupe semi-simple arbitraire, cette conjecture a été prouvée dans [9]. Dans cette Note on montre que, lorsque dim $X=2$ et $G$ est un $X$-groupe réductif isotrope, la conjecture 1.2 se déduit des conjectures de pureté 1.3 et 1.4 énoncées plus bas. On prouve aussi les conjectures 1.3 et 1.2 lorsque $G$ est quasi déployé et $X$ de dimension 2.

Par la suite on note $R$ un anneau local régulier noethérien, de dimension $d \geq 1$, m son idéal maximal et $K$ son corps de fractions. On choisit un élément $u \in m - m^2$. On écrit $Q=R[u^{-1}]$, $X=\text{Spec } R$, $Y=\text{Spec } Q$. Soient $G$ un $R$-groupe réductif et $G_{\text{ad}}$ le $R$-groupe dérivé de $G$ (la partie semi-simple de $G$). Lorsque $R$-rang $(G_{\text{ad}}) \geq 1$, soient $\mathcal{P}(R)$ l’ensemble des sous-groupes paraboliques minimaux de $G$ sur $R$, $R_u(P)$ le radical unipotent de $P \in \mathcal{P}(R)$ et $G^a(Q)$ le sous-groupe de $G(Q)$ engendré par les sous-groupes $R_u(P)(Q)$, pour $P \in \mathcal{P}(R)$.

Conjecture 1.3 (Pureté). — $H^1(Q_{\text{zar}}, G)=0$.

Conjecture 1.4 ($K_1$-pureté). — Soit $R$-rang $(G_{\text{ad}}) \geq 1$ et soit $P$ un sous-groupe parabolique minimal de $G$ sur $R$. Alors $G(Q)=G^a(Q)P(Q)$.

La conjecture 1.3 généralise une conjecture (2) formulée par Quillen pour $G=\text{GL}_n$ dans [11] et prouvée par Gabber pour $G=\text{GL}_n$ et $\text{PGL}_n$ en dimension $\leq 3$, cf. [13].

Note présentée par Jean-Pierre Serre.
1. INTRODUCTION. — This Note is a continuation of [9] and we shall keep here the terminology and the notations of [9].

Let $X$ be an integral regular noetherian scheme, $K=R(X)$ the field of rational functions on $X$, and $G$ a reductive group scheme over $X$. Consider the following sequence of the cohomology of $G$:

$$(1.1) \quad 1 \to H^1(X_{zar}, G) \to H^1(X_{et}, G) \to H^1(K, G).$$

**Conjecture 1.2** (Serre [12], Grothendieck [12], [5]). — Sequence (1.1) is exact.

In the cases when $X$ is one-dimensional or a local henselian, and $G$ is an arbitrary semisimple $X$-group, the Conjecture has been proved in [9]. In this Note we show that when $\dim X=2$ and $G$ is a reductive isotropic $X$-group Conjecture 1.2 follows from purity Conjectures 1.3 and 1.4 formulated below.

Everywhere below $R$ will be a local regular noetherian ring of dimension $d \geq 1$, $m$ the maximal ideal of $R$ and $K$ the quotient field of $R$. Choose an element $u \in m - m^2$ and denote $Q=R[u^{-1}]$, $X=\text{Spec } R$, $G_u$ the derived $R$-group of $G$ (the semisimple part of $G$). If $R$-rank $(G_u) \geq 1$, let $\mathcal{P}(R)$ be the set of all minimal parabolic $R$-subgroups of $G$ over $R$, $R_u(P)$ the unipotent radical of $P \in \mathcal{P}(R)$, and $G^*(Q)$ the subgroup of $G(Q)$ generated by all subgroups $R_u(P)(Q)$, for $P \in \mathcal{P}(R)$.

**Conjecture 1.3** (Purity). — $H^1(Q_{zar}, G)=0$.

**Conjecture 1.4** ($K_1$-purity). — If $R$-rank $(G_u) \geq 1$ and $P$ a minimal parabolic subgroup of $G$ over $R$, then $G(Q)=G^*(Q)P(Q)$.

Conjecture 1.3 generalizes Conjecture (2) formulated by Quillen for $G=GL_n$ in [11] and proved by Gabber for $G=GL_n$ and $PGL_n$ and $X$ of dimension $\leq 3$, cf. [13].

We prove here Conjectures 1.2 and 1.3 and a weak version of Conjecture 1.4, sufficient for our purpose, in the case when $\dim X=2$ and $G$ is quasi-split over $X$. Moreover, we establish in this case several decompositions of the groups $G(Q)$, $G(K)$ and related groups, in particular, versions of the Iwasawa and the Bruhat-Steinberg decompositions for $G(K)$ and $G(Q)$ respectively.

2. LOCALIZATION MAPS AND "LOCAL CLASS SETS". — Let $b=uR$ be the ideal generated by $u$ in $R$. Denote by $\hat{R}$ (resp. $R^\circ$) the completion (resp. henselization) of $R$ with respect to $b$. Put $\hat{Q}=\hat{R}[u^{-1}]$, $Q^\circ=R[u^{-1}]$, $X^\circ=\text{Spec } R^\circ$ and $Y^\circ=\text{Spec } Q^\circ$. Consider the "local class set" $c(G)=G(Q)/G(\hat{Q})G(\hat{R})$ and "henselian local class set" $c^\circ(G)=G(Q)/G(Q^\circ)G(R^\circ)$ which have distinguished points corresponding to the classes $G(Q)G(\hat{R})$ and $G(Q)G(R^\circ)$ respectively.

**Proposition 2.1.** — There exists an exact sequence of pointed sets:

$$(2.1.1) \quad 1 \to c(G) \to H^1(R_{zar}, G) \to H^1(Q_{zar}, G) \times H^1(\hat{R}_{zar}, G).$$

**Proof.** — First, we establish the henselian analogue $(2.1.1)^h$ of $(2.1.1)$ in which $c(G)$ is replaced by $c^h(G)$ and $G(\hat{R})$ by $G(R^\circ)$. $(2.1.1)^h$ is proved by comparing the local cohomology exact sequences for the pairs $(X, Y)$ and $(X^\circ, Y^\circ)$ and using the excision for $H^1(\hat{Q}, Z)$ where $Z=\text{Spec } R/b$. Then we show using the Artin approximation that $(2.1.1)^h$ implies $(2.1.1)$.

Let $m(G): H^1(R_{zar}, G) \to H^1(Q_{zar}, G)$ and $m(G \otimes_R \hat{R}): H^1(\hat{R}_{zar}, G) \to H^1(\hat{Q}_{zar}, G)$ be the canonical maps.

**Corollary 2.2.** — Assume that $\text{Ker } m(G \otimes_R \hat{R})=0$. 

Then the following properties (1), (2) are equivalent:
(1) \(c(G) = 0\);
(2) \(\text{Ker } m(G) = 0\).

**Proposition 2.3.** Let \(M\) be an \(R\)-group of the multiplicative type. Then
(1) the localization maps \(l(M)\) and \(l(M \otimes_R Q)\) are injective;
(2) \(c(M) = 0\), i.e. \(M(Q) = M(Q)M(\hat{R})\), and \(H^1(Q_{zar}, M) = 0\).

The injectivity of \(l(M)\) is known [3], [8], and its proof can be extended to \(l(M \otimes_R Q)\) (see [1] in the case when \(M = G_n\)). The vanishing of \(c(M)\) [resp. \(H^1(Q_{zar}, M)\)] follows from (1) and 2.2 (resp. from Conjecture 1.2 for \(X = \text{Spec } Q\) proved in [9], and (1)).

3. An approximation property of \(G(Q)\). Beginning from this section and up to the end of this Note we shall assume that the residue field \(k = R/m\) of \(R\) is infinite.

Equip \(\hat{Q}\) with the \(\hat{b}\)-adic topology as an \(\hat{R}\)-module, where \(\hat{b} = u \hat{R}\). This uniquely determines the structure of a topological group on \(G(\hat{Q})\). For a subgroup \(H \subset G(\hat{Q})\) denote by \(\hat{H}\) its closure in \(G(\hat{Q})\).

**Proposition 3.1.** \(G(Q)\) contains a subgroup \(N\) which is open in \(G(\hat{Q})\) and is normalized by \(G(\hat{R})\).

In the case where \(\dim R = 1\) and, hence, \(Q = K\) is a field, an analogue of Proposition 3.1 for the pair \((G(K), G(K))\) has been proved by Harder [6]. Our proof of 3.1 is based on similar ideas and uses heavily the results of [4] on the local structure of the \(R\)-scheme \(\mathcal{F}\) of maximal tori of \(G\) over local rings.

4. Some decompositions of \(G(\hat{Q})\).

**Lemma 4.1.** Let \(T\) be an \(\hat{R}\)-torus, \(P\) a parabolic subgroup of \(G\) over \(\hat{R}\), and \(U = R_u(P)\). Then \(U(Q) = G(Q), G^*(\hat{Q}) \subset G(\hat{Q})\) and \(T(Q) \subset G(Q)\).

Let \(R_1 = \hat{R}_1\) be the localization of \(\hat{R}\) with respect to \(\hat{b}\), \(K_1\) the field of fractions of \(R_1\), \(b_1 = u R_1\). Notice that \(\dim R_1 = 1\). Denote by \(\hat{R}_1\) and \(\hat{K}_1\) the \(\hat{b}_1\)-adic completions of \(R_1\) and \(K_1\) respectively.

**Lemma 4.2.** Let \(P\) be a parabolic subgroup of \(G\) over \(\hat{R}\), which is minimal over \(K_1\). Then \(P(\hat{Q}) = G(\hat{Q})G(\hat{R})\).

Lemmas 4.1, 4.2 are generalizations of one-dimensional results of [9], and their proofs follow the same general pattern and use 3.1, 2.3, the local structure of the \(R\)-scheme \(\mathcal{F}\) of maximal tori of \(G\) in [4], and some facts from the Bruhat-Tits theory [2], applied to \(G(\hat{K}_1)\), as key ingredients. Combining 4.1, 4.2 and 5.2 (2) below, we obtain:

**Lemma 4.3.** Assume that \(\dim R = 2\) and that \(G(Q) = G(\hat{Q})P(\hat{Q})\) for a parabolic \(\hat{R}\)-subgroup \(P\) of \(G\) minimal over \(\hat{R}\). Then \(G(\hat{Q}) = G(Q)G(\hat{R})\).

Let \(S\) be a maximal \(R\)-split \(R\)-torus of \(G\), \(\Phi = \Phi(G)\) [resp. \(\Delta = \Delta(G)\)] the set of all (resp. all simple) \(R\)-roots of \(G\), \(U_\alpha\) the unipotent root \(R\)-subgroup of \(G\) corresponding to \(\alpha \in \Phi\), \(G_\alpha\) the \(R\)-subgroup of \(G\) generated by all \(U_\beta\) with \(\beta = k \alpha, k = \pm 1, \pm 2\). Let \(W\) be the Weyl \(R\)-group of \(G\) and \(w = r_{\alpha_1(\omega_1)} r_{\alpha_2(\omega_2)} \cdots r_{\alpha_n(\omega_n)}\) a reduced decomposition of \(w \in W(Q)\) into the product of reflections \(r_{\alpha(\omega)}\) with respect to simple roots \(\alpha(\omega), \alpha \in \Delta^*\). Denote by \(G^1(Q)\) the subgroup of \(G(Q)\) generated by all \(G_\alpha(Q), \alpha \in \Delta^*\).
Theorem 4.4. — Assume that \( \dim R = 2 \) and that \( G \) has a Borel subgroup \( B \) over \( R \). Denote \( B_n = B \times \circ \Gamma \). Then, for all \( \alpha \in \Delta^* \):

1. \( G_\alpha(K) = G_\alpha(Q) \times B_\alpha(K) \), and \( G(K) = G(Q) \times B(K) \).
2. There exists a system of representatives \( Y_\alpha(Q) \) of \( B_\alpha(K) \setminus B_\alpha(K) \cdot r_\alpha B_\alpha(K) \) in \( G_\alpha(Q) \), and the group \( G(Q) \) has a Bruhat-Steinberg decomposition

\[
G(Q) = \bigcup_{\omega \in \Omega} B(Q) Y_{\alpha_{\omega(1)}}(Q) Y_{\alpha_{\omega(2)}}(Q) \cdots Y_{\alpha_{\omega(\ell)}}(Q).
\]

Proof. — General results in [10] reduce the proof of Theorem 4.4 to the vanishing of \( H^1(Q_{2n}, B_n) \) which follows from 2.3 (2).

Corollary 4.5. — Assume that \( \dim R = 2 \) and that \( G \) has a Borel subgroup \( B \) over \( R \). Then \( G(Q) = G^1(Q) \times B(Q) \).

Proposition 4.6. — Let \( R \), \( G \) and \( B \) be such as in 4.5. Then \( G(Q) = G(Q) \times B(Q) \) and \( G(Q) = G(Q) \times G(R) \). If \( G_\alpha \) splits over \( R \), then \( G(Q) = G(Q) \times B(Q) \).

Proof. — We establish, first, the decompositions of 4.6 for \( G_\alpha \) and \( B_n = B \times \circ \Gamma \), for \( \alpha \in \Delta(G) \) or for \( \alpha \in \Delta(G \otimes \hat{R}) \), using the classification of quasi-split simple \( R \)-groups of \( R \)-rank 1, and the equality \( SL_2(Q) = SL_2(Q) \) [7]. We show then that if \( G_\alpha \) is simply connected, \( B_\alpha(Q) \subset G(Q) \) and, hence, \( G_\alpha(Q) \subset G(Q) \). These facts together with 4.5 and 4.1 imply 4.6.

5. Some properties of parabolic subgroups of \( G \). — In sections 5, 6 we shall assume that \( \dim R = 2 \) and (as in sections 3, 4) that the residue field of \( R \) is infinite.

Proposition 5.1. — (1) Let \( P \) be a parabolic subgroup of \( G \) over \( Q \). Then the canonical map \( \text{Ker } l(P) \rightarrow \text{Ker } l(G \otimes Q) \) is surjective.

(2) If \( G \) is quasi-split over \( R \), then \( \text{Ker } l(G \otimes Q) = 0 \).

Proof. — Statement (1) is true for an arbitrary Dedekind ring \( D \) and a reductive \( D \)-group \( G \) [8]. Statement (2) follows from (1), applied to a Borel \( R \)-subgroup of \( G \), and 2.3 (1).

Proposition 5.2. — (1) Let \( P \) be a parabolic subgroup of \( G \) over \( \hat{R} \). Then there exist \( \mathfrak{g} \in G(\hat{R}) \) and a parabolic subgroup \( P' \) of \( G \) over \( \hat{R} \) such that \( P' \otimes \hat{R} = P \mathfrak{g} P^{-1} \). In particular, if \( G \) is quasi-split over \( \hat{R} \), it is quasi-split over \( \hat{R} \).

(2) Let \( P \) be a minimal parabolic subgroup of \( G \) over \( \hat{R} \). Then \( P' \otimes \hat{R} \) is a minimal parabolic subgroup of \( G \) over \( \hat{R} \).

The proof uses the smoothness and the projectivity of the scheme \( \mathfrak{P} \) of parabolic subgroups of \( G \) [4].

6. Reduction and proof of Conjecture 1.2 (\( \dim R = 2 \)). — Let \( K \) be the field of fractions of \( R \). Denote \( \hat{R} = R/b \), \( \hat{K} = R/b \). Since \( \dim R = 2 \), \( \hat{R} \) is a discrete valuation ring and \( \hat{K} \) is the field of fractions of \( \hat{R} \).

Proposition 6.1. — Assume that Conjecture 1.3 is true for \( G \), and that \( G \) has a minimal proper parabolic subgroup \( P \) over \( \hat{K} \) for which \( G(Q) = G(Q) \times P(Q) \).

Then Conjecture 1.2 is true for \( G \).

Remark 6.2. — If Conjecture 1.4 holds for \( G \otimes R \), then \( G(Q) = G(Q) \times P(Q) \) by 4.1.

Proof. — Notice, first, that since the rings \( \hat{R} \) and \( \hat{R} \) are complete and \( G \) is smooth over \( R \), the natural maps \( i : H^1(\hat{R}, G) \rightarrow H^1(\hat{R}, G) \) and \( i : H^1(\hat{R}, G) \rightarrow H^1(\hat{R}, G) \) are bijections [4]. It has been proved in [9] that for the discrete valuation rings \( \hat{R} \) and \( \hat{R} \), \( \text{Ker } l(G \otimes \hat{R}) = \text{Ker } l(G \otimes \hat{R}) = 0 \). It follows from these facts that the
map \( \alpha(G) : H^1(\hat{\mathbb{R}}_m, G) \rightarrow H^1(\hat{\mathbb{R}}_{1,m}, G) \) has trivial kernel. The map \( m(G \otimes_{\mathbb{R}} \hat{\mathbb{R}}) : H^1(\hat{\mathbb{R}}_m, G) \rightarrow H^1(\hat{\mathbb{Q}}_m, G) \) factorizes through the composition \( l(G \otimes_{\mathbb{R}} \hat{\mathbb{R}}) \circ \alpha(G) \) and, hence, also has trivial kernel. Notice, that under the assumptions of 6.1, \( c(G) = 0 \) (prop. 4.3), and \( \text{Ker} l(G \otimes_{\mathbb{R}} \hat{\mathbb{Q}}) = 0 \) (Conjecture 1.3 for \( G \)). The triviality of \( \text{Ker} m(G \otimes_{\mathbb{R}} \hat{\mathbb{R}}) \) and \( c(G) \) implies that \( \text{Ker} l(G) = 0 \) (prop. 2.2). Since \( l(G) = l(G \otimes_{\mathbb{R}} \hat{\mathbb{Q}}) \circ m(G) \), we conclude that \( \text{Ker} l(G) = 0 \).

**Theorem 6.3.** — Let \( G \) be a reductive \( \mathbb{R} \)-group quasi-split over \( \mathbb{R} \).

Then Conjecture 1.2 is true for \( G \).

**Proof.** — The assumptions of 6.1 for such \( G \) are satisfied by 5.1 (2) and 4.6.

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**References**


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Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, U.S.A.