A modular version of the Erdős-Szekeres theorem

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Abstract

Bialostocki, Dierker, and Voxman proved that for any $n \geq p + 2$, there is an integer $B(n, p)$ with the following property. Every set of $B(n, p)$ points in general position in the plane has $n$ points in convex position such that the number of points in the interior of their convex hull is $0 \mod p$. They conjectured that the same is true for all pairs $n \geq 3$, $p \geq 2$. In this note, we show that every sufficiently large point set determining no triangle with more than one point in its interior has $n$ elements that form the vertex set of an empty convex $n$-gon. As a consequence, we show that the above conjecture is true for all $n \geq 5p/6 + O(1)$.

1 Introduction

We say that a set of points in the plane is in general position if no three of them are collinear. Throughout this paper, $\mathcal{X}$ will denote a set of points in the plane in general position. Let $\text{vert}(\mathcal{X})$ denote the vertex set of the convex hull of $\mathcal{X}$. A polygon is said to be empty, if it contains no elements of $\mathcal{X}$ in its interior. If every triple in $\text{vert}(\mathcal{X})$ determines an empty triangle, then $\mathcal{X} = \text{vert}(\mathcal{X})$ is in convex position or, in short, convex.

According to a well known theorem of Erdős and Szekeres [ES1, ES2], for any integer $n \geq 3$, there exists $E(n) = O(4^n)$ with the property that every set $\mathcal{X}$ of at least $E(n)$ points in general position in the plane has $n$ elements in convex position. (In this case, we say that $\mathcal{X}$ determines a convex $n$-gon.) For a long time it appeared to be only a technicality that none of the existing proofs yielded the stronger result that every sufficiently large point set contains the vertex set of an empty convex $n$-gon. Harborth

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[Ha] showed that every 10-element point set determines an empty convex \textit{pentagon}, and that this does not remain true for all 9-element sets. Finally, in 1983 Horton [Ho] surprised most experts by a simple recursive construction of arbitrarily large finite point sets determining no empty convex \textit{heptagons}. The corresponding problem for \textit{hexagons} is still open.

Bialostocki, Dierker, and Voxman [BDV] proposed the following elegant “modular” version of the original problem.

**Conjecture.** For any \( n \geq 3 \) and \( p \geq 2 \), there exists an integer \( B(n, p) \) such that every set of \( B(n, p) \) points in general position in the plane determines a convex \( n \)-gon such that the number of points in its interior is \( 0 \mod p \).

Bialostocki et al. verified this conjecture for every \( n \geq p + 2 \). The original upper bound on \( B(n, p) \) was later improved by Caro [C], but his proof also relied heavily on the assumption \( n \geq p + 2 \).

In the present note we somewhat relax this condition.

**Theorem 1.** For any \( n \geq 5p/6 + O(1) \), there exists an integer \( B(n, p) \) such that every set of \( B(n, p) \) points in general position in the plane determines a convex \( n \)-gon such that the number of points in its interior is \( 0 \mod p \).

If every triple in \( \text{vert}(\mathcal{X}) \) determines a triangle with at most one point in its interior, then \( \mathcal{X} \) is said to be \textit{almost convex}.

Our proof of Theorem 1 is based on the following

**Theorem 2.** For any \( n \geq 3 \), there exists an integer \( K(n) \) such that every almost convex set of at least \( K(n) \) points in general position in the plane determines an empty convex \( n \)-gon. Moreover, we have \( K(n) = \Omega(2^{n/2}) \).

In Sections 2 and 3, we establish Theorems 2 and 1, respectively.

## 2 Almost convex sets – Proof of Theorem 2

Let \( \mathcal{X} \) be a set of points in the plane in general position. For any triple \( x, y, z \in \mathcal{X} \), let \( \triangle xyz \) stand for the triangle determined by \( x, y, z \). Let \( \text{conv}(\mathcal{X}) \) denote the convex hull of \( \mathcal{X} \). Given any convex polygon \( C \), let \( \text{int}(C) \) denote the interior of \( C \).

First, we rephrase the definition of almost convexity. Let \( \mathcal{X} \) denote a set of \( n \) points in the plane in general position.

**Lemma 2.1.** \( \mathcal{X} \) is almost convex if and only if at least one of the following two conditions is satisfied.

(i) Every triangle determined by \( \mathcal{X} \) contains at most one point of \( \mathcal{X} \) in its interior.

(ii) For every subset \( \mathcal{Y} \subseteq \mathcal{X} \) with \( |\mathcal{Y}| \geq 3 \), we have \( |\text{vert}(\mathcal{Y})| \geq |\mathcal{Y}|/2 | + 1 \).

**Proof:** To prove part (i), let \( x, y, z \in \mathcal{X} \), and assume that none of these points lie on the boundary of \( \text{conv}(\mathcal{X}) \). (The other cases can be settled analogously.) Let \( u_1, u_2 \) be the intersection points of the line \( xy \) with the boundary of \( \text{conv}(\mathcal{X}) \), and let \( z_i z_i' \) be the edge of \( \text{conv}(\mathcal{X}) \) such that \( u_i \in z_i z_i' \)
(i = 1, 2). There is an edge $z_3 z'_3$ of $\conv(X)$ such that the $\triangle z_1 z_3 z'_3$ contains $z$. Consequently, $C = \conv(\{z_1, z_2, z_3, z'_1, z'_2, z'_3\}) \supseteq \triangle xyz$. Since $X$ is almost convex, $\text{int}(C)$ contains at most 4 points of $X$, so there cannot be more than one point of $X$ in the interior of $\triangle xyz$.

Next we prove that every almost convex set $X$ satisfies condition (ii). Suppose that a subset $Y$ of $X$ contains at least 3 points. It follows from part (i) that $Y$ is almost convex. Consequently, $|\text{vert}(Y)| \geq \lceil |Y|/2 \rceil + 1$, as required. On the other hand, if the convex hull of every 5-element subset of $X$ has at least 4 vertices, then $X$ is almost convex.

Part (ii) of Lemma 2.1 immediately implies

**Corollary 2.2.** Every subset of an almost convex set is almost convex.

We need the following recursive construction. Let $R_1$ be a set of two points in the plane. Assume that we have already defined $R_1, \ldots, R_j$ so that

1. $X_j := R_1 \cup \ldots \cup R_j$ is in general position,
2. the vertex set of the polygon $\Gamma_j := \conv(X_j)$ is $R_j$, and
3. any triangle determined by $R_j$ contains precisely one point of $X_j$ in its interior.

Let $z_1, z_2, \ldots, z_r$ denote the vertices of $\Gamma_j$ in clockwise order, and let $\varepsilon_j, \delta_j > 0$. For any $1 \leq i \leq r$, let $\ell_i$ denote the line through $z_i$ orthogonal to the bisector of the angle of $\Gamma_j$ at $z_i$. Let $z'_i$ and $z''_i$ be two points on $\ell_i$, at distance $\varepsilon_i$ from $z_i$. Finally, move $z'_i$ and $z''_i$ away from $\Gamma_j$ by a distance $\delta_j$, in the direction orthogonal to $\ell_i$, and denote the resulting points by $u'_i$ and $u''_i$, respectively. (See Fig. 1.)

![Figure 1](image_url)

It is easy to see that if $\varepsilon_j$ and $\delta_j/\varepsilon_j$ are sufficiently small, then $R_{j+1} := \{u'_i, u''_i \mid i = 1, 2, \ldots, r\}$ also satisfies the above three conditions.
We have to verify only the last condition. If \( a \in \{ u'_i, u''_i \} \), \( b \in \{ u'_i, u''_i \} \), and \( c \in \{ u'_k, u''_k \} \) are three points of \( \mathcal{R}_{j+1} \), for three distinct indices \( i, j, k \), then any point of \( \mathcal{X}_{j+1} = \mathcal{X}_j \cup \mathcal{R}_{j+1} \) which belongs to the interior of \( \triangle abc \) must coincide with the unique point of \( \mathcal{X}_j \) in the interior of \( \triangle z_iz_jz_k \). If there exist \( i \neq k \) such that \( a = u'_i, b = u''_i, \) and \( c \in \{ u'_k, u''_k \} \), then the only point of \( \mathcal{X}_{j+1} \) inside \( \triangle abc \) is \( z_i \).

Obviously, we have \( |\mathcal{X}_k| = 2^{k+1} - 2 \) for every \( k \geq 1 \). Since no three vertices of an empty convex polygon determined by \( \mathcal{X}_k \) belong to the same \( \mathcal{R}_i \), it follows that any such polygon has at most \( 2k \) vertices. Consequently, if \( K(n) \) exists, its order of magnitude is at least \( 2^n/2 \).

Next we prove the existence of \( K(n) \).

In the sequel, we use the following notation. For any subset \( \mathcal{Y} \subseteq \mathcal{X} \), let \( \mathcal{Y}' \) denote the set of all points of \( \mathcal{X} \) belonging to the interior of the convex hull of \( \mathcal{Y} \).

**Lemma 2.3.** Suppose that \( \mathcal{R}_1, \ldots, \mathcal{R}_k \subseteq \mathcal{X} \) are in general position in the plane, and they satisfy the following conditions:

(i) \( |\mathcal{R}_1| \geq 2 \);
(ii) \( \mathcal{R}_j \) is in convex position, for \( 1 \leq j \leq k \);
(iii) every triangle of \( \mathcal{R}_j \), \( 1 \leq j \leq k \), has precisely one point of \( \mathcal{X} \) in its interior;
(iv) \( \mathcal{R}_{j+1} = \operatorname{vert}(\mathcal{R}_j') = \operatorname{vert}(\operatorname{int}(\operatorname{conv}(\mathcal{R}_j) \cap \mathcal{X})) \), for every \( 1 < j \leq k \).

Then we have

(a) \( |\mathcal{R}_{j+1}| = 2|\mathcal{R}_j| \), for every \( 1 \leq j \leq k - 1 \).

(b) If \( z_1, \ldots, z_r \) denote the vertices of \( \mathcal{R}_j \) in clockwise order, then the vertices of \( \mathcal{R}_{j+1} \) can be labeled in clockwise order by \( c(z_1), d(z_1), \ldots, c(z_r), d(z_r) \) such that every \( z_i \) \((1 \leq i \leq r)\) lies in the intersection of \( \triangle d(z_{i-1})c(z_i)d(z_i) \) and \( \triangle c(z_i)d(z_i)c(z_{i+1}) \), where the indices are taken modulo \( r \).

(c) \( \mathcal{X} \) determines an empty convex \( 2k \)-gon.

**Proof:** It follows from the properties of the sets \( \mathcal{R}_j \) that \( |\mathcal{R}_j'| = |\mathcal{R}_j| - 2 \) and \( |\mathcal{R}_{j+1}'| = |\mathcal{R}_{j+1}| - 2 \), for every \( 1 \leq j < k \). We also have that

\[
|\mathcal{R}_{j+1}'| = |\mathcal{R}_j| - |\mathcal{R}_j'|
\]

which proves part (a).

To establish part (b), denote by \( u_1, u_2, \ldots, u_{2r} \) the vertices of \( \mathcal{R}_{j+1} \) in clockwise order. Consider the triangles \( T_i = \triangle u_i u_{i+1} u_{i+2} \), for \( 1 \leq i \leq 2r \). Each triangle \( T_i \) contains exactly one point of \( \mathcal{X} \), and it must belong to \( \mathcal{R}_j \). Since \( T_1, T_3, \ldots, T_{2r-1} \) are openly disjoint, each point of the \( r \)-element set \( \mathcal{R}_j \) must lie in one of them. The same is true for \( T_2, T_4, \ldots, T_{2r} \). Thus, there are only two possibilities: each of the regions \( T_1 \cap T_2, T_3 \cap T_4, \ldots, T_{2r-1} \cap T_{2r} \) contains precisely one point of \( \mathcal{R}_j \), or each of \( T_2 \cap T_3, T_4 \cap T_5, \ldots, T_{2r} \cap T_1 \) contains exactly one point of \( \mathcal{R}_j \). In either case we are done.

Finally, we prove part (c). Let \( x_1 \) and \( y_1 \) denote two consecutive vertices of \( \mathcal{R}_1 \) in the clockwise order. Using the notation in part (b), let \( x_{j+1} := d(x_j) \) and \( y_{j+1} := c(y_j) \), for \( j = 1, \ldots, k - 1 \). We show that \( x_1, x_2, \ldots, x_k, y_k, y_{k-1}, \ldots, y_1 \) in this order, induce an empty convex polygon.

For every \( 1 \leq j < k \), \( x_j \) and \( y_j \) lie inside the polygon \( \operatorname{conv}(\mathcal{R}_{j+1}) \), whose 4 consecutive vertices are \( c(x_j), d(x_j) = x_{j+1}, c(y_j) = y_{j+1}, \) and \( d(y_j) \). It follows from part (b) that the line \( x_jy_j \) intersects sides \( c(x_j)d(x_j) \) and \( c(y_j)d(y_j) \) of this polygon. Thus, \( D_j = x_jx_{j+1}y_{j+1}y_j \) is a convex quadrilateral.
Furthermore, the line $x_jy_j$ separates $x_{j+1}y_{j+1}$ from $\mathcal{R}_{j+1}'$, and $D_j$ is empty. To complete the proof, it suffices to check that these quadrilaterals fit together appropriately. That is, for $1 < j < k$, the angles $\alpha_j = \angle x_{j-1}x_jy_j + \angle y_jx_jx_{j+1}$ and $\beta_j = \angle y_jx_{j-1}y_{j+1} + \angle x_jy_{j+1}x_{j+1}$ are smaller than $\pi$. To see that $\alpha_j < \pi$, notice that it follows from part (b) that both lines $d(c(x_{j-1}))x_{j+1}$ and $c(x_j)y_{j+1}$ separate $x_j$ from $x_{j-1}$. Consequently, $x_j$ lies inside $\triangle x_{j-1}x_{j+1}c(x_j)$, so $x_{j-1}, y_{j+1}$, and $y_j$ are on the same side of the line $x_jx_{j+1}$. The other inequality can be checked analogously.  

**Lemma 2.4.** For any positive integers $n \geq 3$ and $k$, there exists $L(n, k)$ such that every almost convex set $\mathcal{X}$ of at least $L(n, k)$ points contains either an empty convex $n$-gon, or a sequence of subsets $\mathcal{R}_1, \ldots, \mathcal{R}_k$ satisfying conditions (i)-(iv) in Lemma 2.3.

Suppose for a moment that we have already established Lemma 2.4. Now we can prove Theorem 2 as follows.

Let $K(n) = L(n, [n/2])$, and let $\mathcal{X}$ be an almost convex set whose size is at least $K(n)$. By Lemma 2.4, $\mathcal{X}$ either contains an empty convex $n$-gon, and we are done, or it has a sequence of subsets $\mathcal{R}_1, \ldots, \mathcal{R}_k$ $(k = [n/2])$ satisfying conditions (i)-(iv). In the latter case, Lemma 2.3(c) guarantees the existence of an empty $n$-gon or $(n+1)$-gon, depending on the parity of $n$. This completes the proof of Theorem 2.

It remains to verify Lemma 2.4.

By Ramsey's theorem, there exists a smallest integer $r = r_3(n, m)$ with the following property. For any 2-coloring of the edges of a complete 3-uniform hypergraph of at least $r$ vertices, there is either a set of $n$ vertices, all of whose triples are colored with the first color, or a set of $m$ vertices, all of whose triples are colored with the second color.

Let $n_1 = 2$, and for $j = 1, 2, \ldots, k$ define recursively the numbers $n_j := r_3(n, m_j)$ and $m_{j+1} := 2n_j - 1$. Let $L(n, k) = 2n_k - 3$, and consider an almost convex set $\mathcal{X}$ of size at least $L(n, k)$. It follows from Lemma 2.1 (ii) that $\vert \text{vert}(\mathcal{X}) \vert \geq n_k$. The set $\mathcal{X}_k := \text{vert}(\mathcal{X})$ is almost convex. Color every triangle $T$ determined by $\mathcal{X}_k$ with 0 or 1: with the number of points of $\mathcal{X}$ in the interior of $T$. According to the definition of $n_k$, in $\mathcal{X}_k$ we can find either an $n$-element subset, all of whose triples are of color 0, or an $m_k$-element subset, $\mathcal{Y}_k$, all of whose triples are of color 1. In the former case, there is an empty convex $n$-gon. In the latter case, $\mathcal{Y}_k$ is a convex set, all of whose triangles have precisely one point of $\mathcal{X}$ in their interiors.

Using the notation introduced before Lemma 2.3, let $\mathcal{X}_{k-1} := \text{vert}(\mathcal{Y}_k')$. By Corollary 2.2, $\mathcal{X}_{k-1}$ is almost convex, and for any three consecutive vertices of $\text{conv}(\mathcal{Y}_k)$, the unique point of $\mathcal{X}$ in the interior of the triangle determined by them belongs to $\mathcal{X}_{k-1}$. Consequently, we have $\vert \mathcal{X}_{k-1} \vert \geq \vert \mathcal{Y}_k \vert / 2 \geq n_{k-1}$.

Repeating the above procedure with $\mathcal{X}_{k-1}$ in place of $\mathcal{X}_k$, we can find either an empty convex $n$-gon or an $m_{k-1}$-element subset $\mathcal{Y}_{k-1} \subseteq \mathcal{X}_{k-1}$ in convex position, whose every triangle has precisely one point in its interior. Set $\mathcal{X}_{k-2} := \text{vert}(\mathcal{Y}_{k-1}')$, and continue. At some point we either find an empty convex $n$-gon, or, after $k$ repetitions, we obtain a sequence of sets, $\mathcal{X}_k \supseteq \mathcal{Y}_k, \ldots, \mathcal{X}_1 \supseteq \mathcal{Y}_1$, such that for $j = 1, \ldots, k$

(i) $\vert \mathcal{Y}_1 \vert \geq m_1 = 2$;
(ii) $\mathcal{X}_j$ and $\mathcal{Y}_j$ are in convex position;
(iii) every triangle determined by $\mathcal{Y}_j$ has exactly one point of $\mathcal{X}$ in its interior;
(iv) \( X_{j-1} = \text{vert}(Y'_j) \).

Thus, the sets \( Y_j \) have all the properties (i)–(iv) in Lemma 2.4 (and Lemma 2.3) required from \( R_j \), except that instead of the last property we have the somewhat weaker relation \( Y_{j-1} \subseteq \text{vert}(Y'_j) \).

We finish the proof of Lemma 2.4 by recursively constructing a sequence of sets \( R_1 \subseteq Y_1, \ldots, R_k \subseteq Y_k \) meeting the requirements of the lemma. Let \( R_1 = Y_1 \), and assume that for some \( j < k \) we have already found \( R_1, \ldots, R_j \) such that \( R_{i-1} = \text{vert}(R'_i) \) for \( 1 < i \leq j \), i.e., condition (iv) is satisfied. (The other conditions are hereditary: they are satisfied for the sets \( Y_i \), so they automatically hold for \( R_i \).)

The following statement, applied to \( A = R_j \) and \( B = Y_{j+1} \), shows that there exists \( R_{j+1} \subseteq Y_{j+1} \) such that \( R_j = \text{vert}(R'_{j+1}) \). This completes the recursion step and the proof of Lemma 2.4, and hence of Theorem 2.

**Proposition 2.5.** Let \( A \subseteq Y_j \) and \( B \subseteq Y_{j+1} \) satisfy \( A \subseteq \text{vert}(B') \). Then there exists a subset \( C \subseteq B \) such that \( A = \text{vert}(C') \).

**Proof:** Suppose that \( A \neq \text{vert}(B') \), and let \( w \in \text{vert}(B') \setminus A \).

We claim that \( \text{conv}(B) \) has three consecutive vertices, \( a, b, c \), (in this order) such that the triangle determined by them contains \( w \) in its interior.

To verify this claim, observe that any line \( \ell \) through \( w \), tangent to \( \text{conv}(B') \), separates at most two vertices of \( B \) from \( B' \). If \( \ell \) separates precisely one such vertex, then this vertex and the two neighboring vertices determine a triangle which contains \( w \) in its interior. If \( \ell \) separates two such vertices, \( x \) and \( y \), then it is easy to see that one of the triangles \( wxy \) and \( zyw \) must contain \( w \) in its interior, where \( u \) and \( v \) denote the vertices of \( \text{conv}(B) \) immediately preceding and following \( \{x, y\} \), respectively. This proves the claim.

To finish the proof of the lemma, let \( B_1 \) denote the set obtained from \( B \) by deleting the point \( b \) whose existence is guaranteed by the claim. We have that \( B'_1 = B' \setminus \{w\} \), and \( A \subseteq \text{vert}(B'_1) \). Note that \( \text{vert}(B'_1) \) is not necessarily a subset of \( \text{vert}(B') \).

If \( \text{vert}(B'_1) = A \), then \( C := B_1 \) will meet the requirements. Otherwise, repeat the argument with \( B_1 \) in place of \( B \) to obtain a subset \( B_2 \subseteq B_1 \) with \( A \subseteq \text{vert}(B'_2) \), etc. After finitely many steps, this procedure must terminate. \( \square \)

### 3 Proof of Theorem 1

Let \( n \geq 5p/6 + O(1) \), and let \( X \) be a set of \( N \) points in the plane. If \( n > p + 1 \), then the assertion was established in [BDV]. Thus, we may assume that \( n \leq p + 1 \) and that \( p \) is sufficiently large. In fact, it follows from our argument that the theorem holds for \( n \geq 5p/6 + 6 \), provided that \( p \geq 264 \).

By the Erdős-Szekeres Theorem, there exists a subset \( X' \subseteq X \) of \( N' \geq \log_4 N \) points in convex position. Let \( x_1, \ldots, x_{N'} \) denote the points of \( X' \) listed in clockwise order.

**Definition 3.1.** For any set \( C \), let \( \langle C \rangle \) denote the number of points of \( X \) in the interior of the convex hull of \( C \), and let \( \langle C \rangle_p \) denote the same number reduced modulo \( p \).
A convex polygon $C$ is said to be *modulo $p$ empty* or, shortly, *$p$-empty*, if $\langle C \rangle_p = 0$. Given an ordered triple $x_i x_j x_k$ ($i < j < k$) and a point $x \in \mathcal{X}$ in the interior of $T = \triangle x_i x_j x_k$, we say that $x$ is the *lowest point* of $\mathcal{X}$ in $T$ (with respect to its “long” side, $x_i x_k$) if no point of $\mathcal{X}$ in $T$, different from $x_i$ and $x_k$, is closer to the line $x_i x_k$ than $x$ is. (By slightly perturbing the elements of $\mathcal{X}$, if necessary, we can assume that this point is uniquely determined.)

Color the triples $\{x, x', x''\} \subset \mathcal{X}$ with $p + 1$ colors, $0, 1, \ldots p$, according to the following rule.

- $\{x, x', x''\}$ gets color $p$ if $\langle x, x', x'' \rangle = 1$.
- $\{x, x', x''\}$ gets color $1$ if $\langle x, x', x'' \rangle_p = 1$ and $\langle x, x', x'' \rangle \neq 1$.
- For $0 \leq i < p$, $i \neq 1$, $\{x, x', x''\}$ gets color $i$ if $\langle x, x', x'' \rangle_p = i$.

It follows from Ramsey’s Theorem, that there is an $M$-element subset $\mathcal{Y} \subset \mathcal{X}$, $M = \Omega(\log \log \log N)$, all of whose triples are of the same color, say, color $q$. Let $y_1, \ldots, y_M$ be an enumeration of the vertices of $\mathcal{Y}$, in clockwise order.

**Claim 3.2.** If $p$ and $q$ are not relatively prime and $N$ (hence, $M$) is sufficiently large, then $\mathcal{X}$ determines a $p$-empty convex $n$-gon.

**Proof:** Suppose that $(p, q) = d > 1$. Then there exists an integer $s$, $p/2 \leq s \leq 2p/3$ such that $sq \equiv 0 \pmod{p}$.

If $q = p$, then $\mathcal{X} \cap \text{conv}(\mathcal{Y})$ is an almost convex set, whose size is at least $M$, and the result follows from Theorem 2. Otherwise, consider any triangulation of the polygon $P = y_1 y_3 y_5 \ldots y_{2s+3}$. Obviously, $P$ consists of $s$ triangles, so it is $p$-empty. Since $2p/3 + 2 \leq n \leq p + 3$, we have $0 \leq n - s - 2 \leq s + 1$. Thus, for $i = 1, 2, \ldots, n-s-2$, there is a lowest point $w_i \in \mathcal{X}$ in $\triangle y_{2i-1} y_{2i} y_{2i+1}$. Using the fact that, for every $i$, $y_{2i-1} w_i y_{2i+1}$ is an empty triangle, we obtain that $\text{conv}(y_1, y_3, \ldots, y_{2s+3}, w_1, \ldots, w_{n-s-2})$ is a $p$-empty convex $n$-gon.

Thus, we can and will assume in the sequel that $p$ and $q$ are relatively prime.

**Definition 3.3.** For any triangle $T = \triangle y_i y_j y_k$ ($i < j < k$) determined by $\mathcal{Y}$, and for any point $x \in \mathcal{X}$ belonging to $T$, we say that $\triangle y_i x y_k$ is a *base sub-triangle*. It is called *standard* if $\langle y_i, x, y_k \rangle_p = 0$ or $q$.

A convex quadrilateral $y_i x x' y_k$ is called a *base sub-quadrilateral*, if $x, x' \in \mathcal{X}$ lie in the interior of $T$. It is *standard* if $\langle y_i, x, x', y_k \rangle_p \equiv 0$, $q$ or $2q$ mod $p$.

Let $\Phi(T)$ (and $\Gamma(T)$) be defined as the set of all numbers that occur as the remainder of the number points in a base sub-triangle (resp., base sub-quadrilateral) of $T$ upon division by $p$. That is, let

$$\Phi(T) = \{ \langle y_i, x, x' \rangle_p \mid x, x' \in \mathcal{X}, x \in \text{int}(T) \cup \{ x_j \} \} ,$$

$$\Gamma(T) = \{ \langle y_i, x, x', y_k \rangle_p \mid x, x' \in \mathcal{X}, x, x' \in \text{int}(T), y_i x x' y_k \text{ convex} \} .$$

Clearly, $\Phi(T)$ can take at most $2^p$ different “values” (sets), and the same is true for $\Gamma(T)$. Therefore, by Ramsey’s Theorem, we can find a subset $Z \subset \mathcal{Y}$, $Z = \{ z_1, z_2, \ldots, z_K \}$ in clockwise order, such that

$$K = \Omega(\log \log M) = \Omega(\log \log \log \log N)$$

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and the pair \((\Phi(T), \Gamma(T))\) is the same for every triangle \(T = \triangle z_i z_j z_k, \ i < j < k\).

**Claim 3.4.** If any triangle determined by \(Z\) has a non-standard base sub-triangle (hence, all of them do) and \(N\) (hence, \(K\)) is sufficiently large, then \(X\) determines a \(p\)-empty convex \(n\)-gon.

**Proof:** Suppose that there exists a non-standard base sub-triangle \(S\) with \(\langle S \rangle_p = s\), and let \(t, 0 \leq t < p\), denote the unique solution of the congruence \(tq \equiv s \mod p\). Since \(S\) is non-standard, \(s \neq 0\) and \(t \neq 0, 1, 2\). It follows from the choice of the set \(Z\) that in every triangle \(\triangle z_i z_j z_k, \ i < j < k\), there is a point \(x \in X\) such that \(\langle z_i, x, z_k \rangle_p = s\). Letting \(l = p - t\), we clearly have \(1 \leq l < p - 2\). We distinguish two cases.

Case 1: \(1 \leq l < 2p/3\). Since \(n \geq 2p/3 + 3\), we can write \(n - 2 = a(l + 1) + b\), where \(a \geq 1\) and \(0 \leq b < l + 1\) are suitable integers. Clearly, we have \(al + 1 \geq a + b\).

The convex polygon \(z_1 z_2 z_3 \ldots z_{2l+3}\) has \(al + 2\) vertices, so its triangulations consist of \(al\) triangles. For \(i = 1, 2, \ldots, a\), let \(x_i\) be a point of \(X\) in \(\triangle z_{2i-1} z_{2i} z_{2i+1}\) such that \(\langle z_{2i-1}, x_i, z_{2i+1} \rangle_p = s\). For \(i = a + 1, a + 2, \ldots, a + b\), let \(x_i\) be the lowest point of \(X\) in \(\triangle z_{2i-1} z_{2i} z_{2i+1}\) so that \(\triangle z_{2i-1} x_i z_{2i+1}\) is empty.

Then \(P = \text{conv}(z_1, z_2, z_3, \ldots, z_{2l+3}, x_1, x_2, \ldots, x_{a+b})\) is a polygon with \(al + 2 + a + b = n\) vertices, and \(\langle P \rangle \equiv \text{aq} \equiv \text{aq} + atq \equiv aq(l + t) \equiv 0 \mod p\).

Case 2: \(2p/3 < l \leq p - 2\). Since \(n \leq p + 1\), we can write \(p + 2 - n = a(t - 1) - b\), where \(a \geq 1\) and \(0 \leq b < t - 1\) are suitable integers. Then we have \(n = p + 2 - a(t - 1) + b\). Using the fact that \(n \geq 2p/3 + 3\), one can easily check that \(p - at + 1 \geq a + b\).

The convex polygon \(z_1 z_2 z_3 \ldots z_{2(p-at)+3}\) has \(p - at + 2\) vertices, so its triangulations consist of \(p - at\) triangles. For \(i = 1, 2, \ldots, a\), let \(x_i\) be a point of \(X\) in \(\triangle z_{2i-1} z_{2i} z_{2i+1}\) such that \(\langle z_{2i-1}, x_i, z_{2i+1} \rangle_p = s\). For \(i = a + 1, a + 2, \ldots, a + b\), let \(x_i\) be the lowest point of \(X\) in \(\triangle z_{2i-1} z_{2i} z_{2i+1}\) so that \(\triangle z_{2i-1} x_i z_{2i+1}\) is empty.

Then \(P = \text{conv}(z_1, z_2, z_3, \ldots, z_{2(p-at)+3}, x_1, x_2, \ldots, x_{a+b})\) is a polygon with \(p - at + 2 + a + b = n\) vertices, and \(\langle P \rangle \equiv \text{q(p-at)} \equiv \text{aq}(l + t) \equiv 0 \mod p\).

**Claim 3.5.** If any triangle determined by \(Z\) has a non-standard base sub-quadrilateral (hence, all of them do) and \(N\) (hence, \(K\)) is sufficiently large, then \(X\) determines a \(p\)-empty convex \(n\)-gon.

**Proof:** Suppose that there exists a non-standard base sub-quadrilateral \(S\) with \(\langle S \rangle_p = s\), and, as before, let \(t\) denote the unique solution of the congruence \(tq \equiv s \mod p\) in the interval \([0, p]\). Since \(S\) is non-standard, we have \(s \neq 0\) and \(t \neq 0, 1, 2\). It follows that every triangle \(\triangle z_i z_j z_k, \ i < j < k\) contains two points \(x, x' \in X\) such that \(z_i x x' z_k\) is a convex quadrilateral and \(\langle z_i, x, x', z_k \rangle_p = s\). Letting \(l = p - t\), we clearly have \(1 \leq l < p - 3\). We distinguish two cases.

Case 1: \(1 \leq l < 2p/3\). Since \(n \geq 2p/3 + 4\), we can write \(n - 2 = a(l + 2) + b\), where \(a \geq 1\) and \(0 \leq b < l + 2\) are suitable integers. Clearly, we have \(al + 2 \geq a + b\).

The convex polygon \(z_1 z_2 z_3 \ldots z_{2l+3}\) has \(al + 2\) vertices, so its triangulations consist of \(al\) triangles. For \(i = 1, 2, \ldots, a\), let \(x_i\) and \(x'_i\) be two points of \(X\) in \(\triangle z_{2i-1} z_{2i} z_{2i+1}\) such that \(z_{2i-1} x_i x'_i z_{2i+1}\) is convex and \(\langle z_{2i-1}, x_i, x'_i, z_{2i+1} \rangle_p = s\). For \(i = a + 1, a + 2, \ldots, a + b\), let \(x_i\) be the lowest point of \(X\) in \(\triangle z_{2i-1} z_{2i} z_{2i+1}\). More precisely, in the exceptional case of \(a + b = al + 2\), let \(x_{a+b}\) be a point in \(\triangle z_{2a+3} z_k z_1\) such that \(\triangle z_{2a+3} x_{a+b} z_1\) is empty.
Then $P = \text{conv}(z_1, z_3, z_5, \ldots, z_{2l+1}, x_1, x_2, x_2', \ldots, x_a, x_a', x_{a+1}, \ldots, x_{a+b})$ is a convex polygon with $a + 2 + 2a + b = n$ vertices, and $\langle P \rangle = al + as \equiv alq + atq \equiv aq(l + t) \equiv 0 \mod p$.

Case 2: $2p/3 < l \leq p - 3$. Since $n \leq p + 1$, we can write $p + 2 = a(t - 2) - b$, where $a, b > 1$ and $0 \leq b < t - 2$ are suitable integers. Then $n = p + 2 - a(t - 2) + b$. Using the fact that $n \geq 3p/4 + 4$, one can easily check that $p - at + 1 > a + b$.

The convex polygon $z_1 z_3 z_5 \ldots z_{2p-at+1}$ has $p - at + 2$ vertices, so its triangulations consist of $p - at$ triangles. For $i = 1, 2, \ldots, a$, let $x_i$ and $x_i'$ be two points of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$ such that $z_{2i-1} x_i x_i' z_{2i+1}$ is a convex quadrilateral with $\langle z_{2i-1} x_i x_i' z_{2i+1} \rangle_p = s$. For $i = a + 1, a + 2, \ldots, a + b$, let $x_i$ be the lowest point of $\mathcal{X}$ in $\triangle z_{2i-1} z_{2i} z_{2i+1}$.

Then $P = \text{conv}(z_1, z_3, z_5, \ldots, z_{2p-at+1}, x_1, x_2, x_2', \ldots, x_a, x_a', x_{a+1}, \ldots, x_{a+b})$ is a convex polygon with $p - at + 2 + 2a + b = n$ vertices, and $\langle P \rangle \equiv q(p - at) + as \equiv 0 \mod p$.

From now on we assume that all base sub-triangles and base sub-quadrilaterals of the triangles determined by $\mathcal{Z}$ are standard.

**Definition 3.6.** For any triangle $T = \triangle z_i z_j z_k$, $i < j < k$, define a partial order on the points in the interior of $T$ as follows. For $x, y \in T$, $x \prec_T y$ if and only if $\triangle z_i y z_k$ contains $x$. The rank of $y$ is the largest number $a$ for which there exist $x_1, x_2, \ldots, x_a$ in $T$ such that $x_1 \prec_T x_2 \prec_T \cdots \prec_T x_a \prec_T y$.

**Claim 3.7.** Let $T = \triangle z_i z_j z_k$, $i < j < k$.

If $q \neq 1, \frac{p+1}{2}$, then there exist $x_0, x_1, \ldots, x_{q-1}$ in $T$ such that $z_i x_0 x_1 \ldots x_{q-1} z_k$ is an empty convex $(q + 2)$-gon.

If $q = \frac{p+1}{2}$, then there exist $x_0, x_1$ in $T$ such that $z_i x_0 x_1 z_k$ is an empty convex quadrilateral.

**Proof:** Suppose that $q \neq 1$. Let $x_0, x_1, \ldots, x_r$ be the points of rank 0 in the interior of $T$, listed in counter-clockwise order of visibility from $z_j$. It follows from the fact that every base sub-triangle is standard that $r \geq q - 1$. For every $0 \leq l \leq r - 1$, the quadrilateral $z_i x_l x_{l+1} z_k$ is convex and empty.

If, in addition, $q \neq \frac{p+1}{2}$, then there is no base sub-quadrilateral containing precisely one element of $\mathcal{X}$, for such a quadrilateral would be non-standard. Consequently, $z_i x_l x_{l+1} z_k$ is an empty convex pentagon for $0 \leq l \leq q - 3$, and the claim is true.

**Claim 3.8.** Suppose that $K \geq 4p - 1$ and $q = 1$. Then $\mathcal{X}$ determines a $p$-empty convex $n$-gon.

**Proof:** Consider the triangle $T = \triangle z_i z_j z_k$, where $i = 1, j = 2p$ and $k = 4p - 1$. Clearly, $T$ (as any other triangle determined by $\mathcal{Z}$) satisfies $\langle T \rangle_p = 1$ and $\langle T \rangle \neq 1$.

Let $x$ denote any point of rank $r$ in $T$. Since every base sub-triangle is standard, it follows by an easy induction that $\langle z_i, x, z_k \rangle \geq \frac{5p}{2}$ if $r$ is even, and $\langle z_i, x, z_k \rangle \geq \frac{5p + 1}{2}$ if $r$ is odd.

Suppose first that $T$ does not contain a point of rank 4. Then $T$ contains at least $p + 1$ points, all of rank 0, 1, 2 or 3. Let $P_0 := T$. We show how to construct a sequence of convex polygons $P_1, P_2, \ldots, P_s$ satisfying the conditions

(i) $z_j$ and $z_k$ are vertices of $P_t \quad (1 \leq t \leq s)$;
(ii) $P_t$ has at most 6 vertices $\quad (1 \leq t \leq s)$;
(iii) every point of $\mathcal{X}$ in $P_t$ belongs to the closure of $P_{t+1} \quad (0 \leq t \leq s - 1)$;
(iv) $P_s$ is empty.

Suppose that we have already defined $P_t$ for some $t \geq 0$. If $\langle P_t \rangle = 0$, then set $s := t$. Otherwise, construct $P_{t+1} = z_j y_1 \ldots y_r z_k$, where $1 \leq r \leq 4$, as follows. Let $y_1$ be the first point of $\mathcal{X}$ lying in $P_t$, in counter-clockwise order of visibility from $z_j$. Let $\mathcal{T}_1$ denote the set of points of $\mathcal{X}$ lying in $P_t$ but not contained in $\triangle z_j y_1 z_k$.

If $\mathcal{T}_1 = \emptyset$, then letting $r = 1$, $P_{t+1} = z_j y_1 z_k$ meets all the requirements. Otherwise, let $y_2$ be the first point of $\mathcal{T}_1$ in counter-clockwise order of visibility from $y_1$. Clearly, $z_j y_1 y_2 z_k$ is a convex quadrilateral, and the rank of $y_2$ is smaller than that of $y_1$. Let $\mathcal{T}_2$ denote the set of points of $\mathcal{T}_1$ not contained in the quadrilateral $z_j y_1 y_2 z_k$. If $\mathcal{T}_2 = \emptyset$, then letting $r = 2$, $P_{t+1} = z_j y_1 y_2 z_k$ meets all the requirements. Otherwise, let $y_3$ be the first point of $\mathcal{T}_2$ in counter-clockwise order of visibility from $y_2$. Clearly, $z_j y_1 y_2 y_3 z_k$ is a convex pentagon, and the rank of $y_3$ is smaller than that of $y_2$. Finally, let $\mathcal{T}_3$ denote the set of points of $\mathcal{T}_2$ not contained in the pentagon $z_j y_1 y_2 y_3 z_k$. If $\mathcal{T}_3 = \emptyset$, then letting $r = 3$, $P_{t+1} = z_j y_1 y_2 y_3 z_k$ meets all the requirements. Otherwise, let $y_4$ be the first point of $\mathcal{T}_3$ in counter-clockwise order of visibility from $y_3$. Clearly, $z_j y_1 y_2 y_3 y_4 z_k$ is a convex hexagon, and the rank of $y_4$ is smaller than that of $y_3$. Therefore, the rank of $y_4$ is 0, and every point of $\mathcal{T}_3$ is contained in the hexagon $z_j y_1 y_2 y_3 y_4 z_k$, which satisfies all the conditions (i)–(iv).

![Figure 2.](image-url)

Suppose next that $T$ contains a point $x$ of rank 4. Let $x'$ denote the intersection point of the line $z_j x$ and the segment $z_i z_k$. Now $\triangle z_i x' z_k$ contains at least $2p$ points of $\mathcal{X}$. Thus, we may assume without loss of generality that $\triangle x x' z_k$ contains at least $p$ points of $\mathcal{X}$, all of rank 0, 1, 2 or 3. In this case, let
$P_0 := z_jx'z_k$.

In the same way as above, one can construct a sequence of convex polygons $P_1, P_2, \ldots, P_s$ satisfying the conditions

(i) $z_j, z_k,$ and $x$ are vertices of $P_t$ ($1 \leq t \leq s$);
(ii) $P_t$ has at least 4 and at most 7 vertices ($1 \leq t \leq s$);
(iii) every point of $\mathcal{X}$ in $P_t$ belongs to the closure of $P_{t+1}$ ($0 \leq t \leq s-1$);
(iv) $P_s \cap \Delta z_i x z_k$ is empty.

In both cases, it follows from the properties of the polygons $P_t$ that $\langle P_t \rangle > \langle P_{t+1} \rangle \geq \langle P_t \rangle - 4$, for $0 \leq t \leq s - 1$. Furthermore, we have $\langle P_0 \rangle - \langle P_s \rangle \geq p - 4$. Therefore, there exists an integer $1 \leq t' \leq s$ such that $\langle P_{t'} \rangle = 7 + r - n$ for some $0 \leq r \leq 7$. Then $P = \text{conv}(P_{t'}, z_{j+2}, z_{j+4}, \ldots, z_{j+2(n-7-r)})$ is a $p$-empty polygon. Suppose $P_{t'}$ has $7-r'$ vertices for some $0 \leq r' \leq 4$. For $m = 1, 2, \ldots, r+r'$, let $w_m$ be the lowest point of $\mathcal{X}$ in $\Delta z_{j+2m-2}z_{j+2m-1}z_{j+2m}$. Then $\text{conv}(P, w_1, w_2, \ldots, w_{r+r'})$ is a $p$-empty $n$-gon.

**Claim 3.9.** Suppose that $K \geq 4p - 1$ and $2 \leq q \leq 6$. Then $\mathcal{X}$ determines a $p$-empty convex $n$-gon.

**Proof:** Consider the triangle $T = \Delta z_i x z_j$, where $i = 1, j = 2p$ and $k = 4p - 1$. Let $x$ be the point of $\mathcal{X}$ in int($T$), closest to the line $z_i z_j$. Then $\Delta z_i x z_k$ is a standard base sub-triangle, so that $\langle z_i, x, z_k \rangle_p = 0$ or $q$. Since $\langle T \rangle_p = q$, we have $\langle z_j, x, z_k \rangle_p = p - 1$ or $q - 1$. In the first case, choose an integer $1 \leq a \leq p - 1$ such that $aq \equiv 1 \mod p$. In the second case, choose an integer $1 \leq a \leq p - 1$ such that $aq \equiv p - q + 1 \mod p$. In either case, $(p - a + 1)/q \leq a \leq (p(q - 1) + 1)/q$.

The polygon $\text{conv}(z_j, z_{j+2}, z_{j+4}, \ldots, z_{j+2a}, z_k, x)$ has $a+3$ vertices and is $p$-empty. Since $n \geq 5p/6 + 4$, $q \leq 6$ and $p \geq 2k$, we have $a + 3 \leq n \leq p + 1 \leq a(q + 1) + 3$. Thus, there is a non-negative integer $f \leq aq$ such that $f + a + 3 = n$. Note that $q < \frac{p+1}{2}$, so we can apply Claim 3.7 to conclude that, for every $1 \leq m \leq a$, there is a $q$-element subset $U_m \subseteq \mathcal{X}$ in the interior of $\Delta z_{j+2m-2}z_{j+2m-1}z_{j+2m}$, which, together with $z_{j+2m-2}$ and $z_{j+2m}$, forms the vertex set of an empty convex $(q+2)$-gon. Let $U$ be an $f$-element subset of $U_1 \cup \ldots \cup U_a$. Then $\text{conv}(z_j, z_{j+2}, z_{j+4}, \ldots, z_{j+2a}, z_k, x, U)$ is a $p$-empty $n$-gon.

For the rest of the proof, we assume that $K \geq 4p - 1$ and $q \geq 7$. Fix $T = \Delta z_i z_j z_k$, where $i = 1, j = 2p$ and $k = 4p - 1$.

**Claim 3.10.** Suppose that all points of $\mathcal{X}$ in the interior of $T$ have rank 0. Then $\mathcal{X}$ determines a $p$-empty convex $n$-gon.

**Proof:** Let $x_0, x_1, x_2, \ldots, x_s$ denote the points of $\mathcal{X}$ in the interior of $T$, listed in counter-clockwise order of visibility from $z_j$. Clearly, we have $s \geq q - 1$. There is an integer $1 \leq a \leq q - 1$ such that $\left\lceil \frac{ap}{q} \right\rceil + 3 \leq n \leq \left\lceil \frac{(a+1)p}{q} \right\rceil + 2$. Then $n = \left\lceil \frac{ap}{q} \right\rceil + 3 + b$, where $0 \leq b \leq \left\lfloor \frac{p}{q} \right\rfloor < \left\lceil \frac{ap}{q} \right\rceil$. Write $ap \equiv c \mod q$ with $1 \leq c \leq q - 1$.

The convex polygon $P = z_i z_{i+2} \ldots z_{i+2\left\lceil \frac{ap}{q} \right\rceil} z_j x_c$ has $\left\lceil \frac{ap}{q} \right\rceil + 3$ vertices and contains $\langle P \rangle \equiv q \left\lceil \frac{ap}{q} \right\rceil + c \equiv 0 \mod p$ points in its interior. For $l = 1, 2, \ldots, b$, let $y_l$ be the lowest point of $\mathcal{X}$ in $\Delta z_{i+2l-2} z_{i+2l-1} z_{i+2l}$.
with respect to the side $z_{i+2} - 2z_{i+2}$. Then $\text{conv}(z_j, z_{i+2}, \ldots, z_{i+2|ap/q|}, z_j, x_c, y_1, \ldots, y_b) \text{ is a } p\text{-empty convex polygon with } \left\lceil \frac{ap}{q} \right\rceil + 3 + b = n \text{ vertices.}\hspace{1cm} \square$

It remains to prove

**Claim 3.11.** Suppose that there is a point of $X$ in the interior of $T$, whose rank is 1. Then $X$ determines a $p$-empty convex $n$-gon.

**Proof:** Let $x \in X$ be a point of rank 1 in the interior of $T$. Then $\Delta z_i x z_k$ is a standard, non-empty base sub-triangle with at least $q$ points in its interior, all of which have rank 0. Let $x_0, x_1, \ldots, x_r$ denote the points of $X$ in the interior of $\Delta z_i x z_k$, listed in counter-clockwise order of visibility from $z_j$. Suppose that the line $z_j x$ separates $x_0, \ldots, x_t$ from $x_{t+1}, \ldots, x_r$. Since $r \geq q - 1$, we may assume without loss of generality that $t \geq t_0 = \left\lfloor \frac{q}{2} \right\rfloor - 1$.

Letting $s_0 := \langle z_i, x_0, x, z_j \rangle$, we have $\langle z_i, x_m, x, z_j \rangle = s_0 + m$, for $0 \leq m \leq t$. Choose an integer $1 \leq s'_0 \leq p$ satisfying $s'_0 \equiv s_0 \text{ mod } p$. Let $I \subseteq \{1, 2, \ldots, q\}$ be an interval of consecutive integers, defined as follows:

$$I = \begin{cases} 
\{2, 3, \ldots, \left\lfloor \frac{q}{2} \right\rfloor + 2\}, & \text{if } 7 \leq q \leq 11; \\
\{\lceil \frac{q}{3} \rceil - 1, \ldots, \lfloor \frac{5q}{6} \rfloor\}, & \text{if } 12 \leq q \neq (p + 1)/2; \\
\{\lceil \frac{q}{3} \rceil + 1, \ldots, \lfloor \frac{5q}{6} \rfloor + 2\}, & \text{if } q = (p + 1)/2.
\end{cases}$$

In view of the fact that $(p, q) = 1$, we have that $|\{(bp \text{ mod } q \mid b \in I)\}| = |I| \geq \left\lfloor \frac{q}{2} \right\rfloor + 1$. Furthermore, $\{|a \text{ mod } q \mid s'_0 \leq a \leq s'_0 + t_0\}| = t_0 + 1 = \left\lfloor \frac{q}{2} \right\rfloor$. Thus, by the pigeonhole principle, there are integers $a, b$ satisfying $s'_0 \leq a \leq s'_0 + t_0$, $b \in I$ such that $bp \equiv a \text{ mod } q$. Let $a = cq + r$, $0 \leq r < q$. Then $0 \leq c < p/q + 1$ and $bp = Cq + r$, where $C = \lceil bp/q \rceil$. Let $a' = a - s'_0$. Clearly, we have $C - c \geq 0$ and $0 \leq a' \leq t$.

The polygon $P = z_i z_{i+2} z_{i+4} \ldots z_{i+2(C-c)} z_j x x_{a'}$ has $C - c + 4$ vertices and $\langle P \rangle \equiv (C - c)q + s_0 + a' = Cq + r - a + s_0 + a' = \text{bp} \equiv 0 \text{ mod } p$.

By modifying $P$, we will increase the number of vertices to $n$ without changing the number of interior points. For $m = 1, 2, \ldots, C - c$, let $U_m \subseteq \Delta z_i z_{i+2m} - 2z_{i+2m - 1} z_{i+2m}$ denote a set of $q$ points if $q \neq \frac{p + 1}{2}$ and a set of 2 points if $q = \frac{p + 1}{2}$, whose existence is guaranteed in Claim 3.7. Let $U = U_1 \cup \ldots \cup U_{C-c}$. Then we have

$$|U| = \begin{cases} 
\frac{q(C - c)}{2}, & \text{if } q \neq \frac{p + 1}{2}; \\
\frac{(C - c)}{2}, & \text{if } q = \frac{p + 1}{2}.
\end{cases}$$

One can readily check that $C - c + 4 \leq C + 4 \leq 5p/6 + 6 \leq n$. It is sufficient to prove that $|U| \geq n - (C - c + 4)$. Then there exists a $U' \subseteq U$, $|U'| = n - (C - c + 4)$ such that $\text{conv}(P \cup U')$ is a $p$-empty $n$-gon. We distinguish three cases.

**Case 1:** $7 \leq q \leq 11$. In this case, $p \geq 264 \geq 2q(q + 1)$, so that $C - c > p/q - 2 \geq p/(q + 1)$. Note that $q \neq \frac{p + 1}{2}$. Thus, $|U| + C - c = (q + 1)(C - c) \geq p \geq n - 4$, and the statement follows.

**Case 2:** $12 \leq q \neq \frac{p + 1}{2}$. We also have $p \geq 24$, so that $1/3 - 2/q - 1/(q + 1) \geq 2/p$ and $C - c > p/3 - 2p/q - 2 \geq p/(q + 1)$, as in the previous case.

**Case 3:** $q = \frac{p + 1}{2}$. In this case, $c \leq 2$ and $C \geq p/3 + 1$. This implies $|U| + C - c = 3(C - c) \geq p - 3 \geq n - 4$, and we are done. \hspace{1cm} \square

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Note that the condition $n \geq 5p/6 + O(1)$ is heavily used in the proofs of Claims 3.9 and 3.11, and our arguments do not allow to replace it by a weaker bound.

References


