Final Projects

Financial Mathematics and Simulation
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Project 1: Portfolio Management and Stochastic Control
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Project 2: Option Pricing in Stochastic Volatility Models
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Project 3: Monte-Carlo Methods in Finance
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General Instructions: Please be sure to answer each of the questions carefully. When appropriate present your results using carefully labeled graphs and figures. If you have any questions please feel free to ask.

Project 1: Portfolio Management and Stochastic Control

The issue of how to optimally make decisions about the allocation of resources is a common problem in many areas of economics and finance. There are many theories that are useful in guiding one in deciding about how to optimally allocate resources. Applying these ideas in practice of course requires a healthy dose of business intuition and experience. In this project you will explore mathematical techniques which are employed in the context of management of investments to maximize overall return.

Let us consider the general problem of allocation of wealth between two investment opportunities $S_t^{(1)}$ and $S_t^{(2)}$ which have value at time $t$ modeled by the stochastic differential equations:

$$
\begin{align*}
    dS_t^{(1)} &= r_1 S_t^{(1)} dt + \sigma_1 S_t^{(1)} dW_t^{(1)} \\
    dS_t^{(2)} &= r_2 S_t^{(2)} dt + \sigma_2 S_t^{(2)} dW_t^{(2)}
\end{align*}
$$

where $\langle dW_t^{(1)} dW_t^{(2)} \rangle = \rho dt$. The value of a portfolio in which we invest $u$ of our present wealth in the first asset and $(1 - u)$ of our present wealth in the second satisfies the stochastic differential equation:

$$
\begin{align*}
    dZ_t &= u Z_t dS_t^{(1)} + (1 - u) Z_t dS_t^{(2)} \\
         &= (ur_1 Z_t + (1 - u)r_2 Z_t) dt + u \sigma_1 Z_t dW_t^{(1)} + (1 - u) \sigma_2 Z_t dW_t^{(2)} \\
         &= r_3(u) Z_t dt + \sigma_3(u) Z_t dW_t^{(3)}
\end{align*}
$$

where $r_3(u) = ur_1 + (1 - u)r_2$, $\sigma_3^2(u) = u^2 \sigma_1^2 + 2 \rho u(1 - u) \sigma_1 \sigma_2 + (1 - u)^2 \sigma_2^2$ and $W_t^{(3)}$ is a standard Brownian motion. To quantify the payoff at time $T$ from the investment, when the initial investment is $z_0$ at time $t_0$, let

$$
J(z_0, t_0, T) = E^{(z_0, t_0)} \left( \int_{t_0}^{T} g(Z_t, u_t, t) dt + f(Z_T, T) \right).
$$

In the case that $g = 0$, the $J(z_0, t_0, T)$ is the expected utility of the wealth at maturity $T$, as modeled by the function $f(z)$. We shall also consider the case $g \neq 0$ to model costs of the investment strategy, such as transaction costs associated with adjusting the allocations of wealth between the assets.

Given the investments strategies above a natural consideration is to try to determine at each instant in time what portion of the wealth $u = u(z, t)$ should be optimally invested in each of the assets in order to maximize the expected wealth. Let the optimal payoff be denoted by $\Phi(z, t) = \sup_{u \in [0, 1]} \{ J(z, t, u) \}$ when the portfolio has value $z$ at time $t$. It can be shown that the optimal attainable wealth $\Phi$ for this class of controls satisfies the Hamilton-Jacobi-Bellman partial differential equation:

$$
\begin{align*}
    \sup_{u \in [0, 1]} \{ Lu(\Phi(z, t) - g(z, u, t)) \} &= 0 \\
    \Phi(z, T) &= f(z, T).
\end{align*}
$$
For the control $u$, the $L^u$ denotes the infinitesimal generator of the stochastic process given by

$$
L^u\Phi = \frac{\partial \Phi}{\partial t} + r_3(u)z \frac{\partial \Phi}{\partial z} + \frac{1}{2} \sigma_3^2(u)z^2 \frac{\partial^2 \Phi}{\partial z^2}.
$$

In your presentation and report you are to address the following questions:

a) Let us consider the special case in which the second investment opportunity is a risk-free asset (bond), with $\sigma_2 = 0$. Also let the utility function of the wealth be $f(z, T) = z^a$, with $0 < a < 1$, and set $g = 0$. In this case solve the above equations analytically to find an expression for the objective function $\Phi(z, t)$ and the optimal control $u(z, t)$ in terms of the parameters of the stochastic process governing the asset prices.

Hint: Use that the supremum, if achieved for a value of $0 < u^* < 1$, satisfies $\frac{\partial L^u\Phi(z, t)}{\partial u} = 0$. Substitute the expression you obtain for $u^*$ into the equation $L^{u^*}\Phi(z, t) = 0$, the expression for $u^*$ may involve derivatives of the function $\Phi$. Use the resulting PDE to find a solution of the form $\Phi(z, t) = h(t)z^a$ (use the equation to determine $h(t)$ after simplifying the expressions as much as possible).

b) Finding analytic expressions for the optimal control is in general quite challenging suggesting a numerical approach be taken. We now discuss a numerical scheme which can be motivated by intuition about the random process and compare its results with the theory above. Developing robust numerical methods for stochastic controls problems is in general challenging and more rigorous analysis would of course have to be done to ensure the proposed numerical scheme indeed approximates the solution to the control problem.

Let us first consider the change of variable to the stochastic process $X_t = \ln(Z_t e^{-rt})$. By Ito’s Lemma the process $X_t$ satisfies $dX_t = (r_3(u) - r - \frac{1}{2} \sigma_3^2(u)) dt + \sigma_3(u)dW_t^{(3)}$. The continuous time stochastic process $X_t$ can be approximated by a random walk on a regular lattice with grid spacing $\Delta x$ and time discretization $\Delta t$. The random walk model we shall consider will have a probability of $q_1$ of moving up one lattice site, probability $q_2$ to remain at the current lattice site, and probability $q_3$ to move down one lattice site. To obtain a random walk model which behaves roughly like the stochastic process we shall require that the probabilities are such that the first two moments agree with the stochastic process over each time step. For any given choice of $u$ this requires the probabilities $q_1, q_2, q_3$ satisfy:

$$
E(X_{(n+1)\Delta t} - X_{n\Delta t}) = \left( r_3(u) - r - \frac{1}{2} \sigma_3^2(u) \right) \Delta t
$$

$$
\text{Var}(X_{(n+1)\Delta t} - X_{n\Delta t}) = \sigma_3^2(u) \Delta t
$$

$$
q_1 + q_2 + q_3 = 1.
$$

When writing out the expectation and variance explicitly in terms of $q_1, q_2, q_3$ this gives three linear equations.

To find an optimal control for the random walk model we shall employ an inductive approach, falling within the class of methods referred to as Dynamic Programming. Considering any lattice site $x$ one period before maturity $T - \Delta t$, we can determine the optimal allocation $u$ of the investment by maximizing the the expectation of the utility function $f(x, T)$ of the portfolio at
time $T$, under the random walk having probabilities $q_1, q_2, q_3$ determined by the moment conditions. Give the general expression at time $T - \Delta t$ for $J(x, u, T - \Delta t)$ under the random walk model of the investment $u$ when the utility function only depends on $x$, $f(x, T) = f(x)$. Give the general expression for the optimal $u$ at each node $x$ of the lattice at time $T - \Delta t$ by finding the maximum of $J(u)$. Now to apply the idea above recursively, derive a similar expression at each node $x$ at time $T - 2\Delta t$ where instead of maximizing $f(x, T)$ the expectation of the function $\Phi(X_{T-\Delta t}, T - \Delta t) = J(X_{T-\Delta t}, u^*(x, T - \Delta t), T - \Delta t)$ is maximized for the period. Give the expression for $u^*(x, T - 2\Delta t)$ for each node $x$ of the lattice in terms of the $\Phi$ values at time $T - \Delta t$. This last expression provides a recursive means to determine $\Phi$ and $u^*$ at each lattice site.

c) These expressions allow for the optimal control at any time to be determined by backward induction on the lattice. Use the above results to give an expression for $u^*(x, T - n\Delta t)$ at a general time $T - n\Delta t$. What does this expression become in the case that one of the assets is risk-free $\sigma_2 = 0$ and $\rho = 0$? Give an expression for $u^*$ in the limit as $\Delta t \to 0, \Delta x \to 0$ (with $\Delta t \leq \frac{\Delta x^2}{\max\{\sigma_1^2, \sigma_2^2\}}$). How does this compare with the analytic result you obtained in a)?

d) Write a matlab code to compute the optimal control $u^*$ and $\Phi(x, t)$ at each node of the lattice, use the utility function $f(e^{x+rt}, T) = e^{a(x+rt)}$ where $0 < a < 1$. For different numbers of periods $N = 4, 10, 100$, and for specific choice of parameters for the assets $r_1 = 10\%, r_2 = 5\%, \sigma_1 = 30\%, \sigma_2 = 10\%, \rho = 0$ compare this with your analytic results in a). How large does $N, (\Delta t = T/N)$, need to be for this to be a good approximation of the result found in part a), say within 10% of the analytic result?

e) How does the above expression change if we now consider over each period a transaction cost for adjusting the portfolio given by $g(x, u, n\Delta t)\Delta t^2 = -(u(n\Delta t) - u((n - 1)\Delta t))^2$. Implement a matlab code to compute the optimal controls $u$ at each node in this case. Try to derive an analytic expression for a general period using the inductive formula (this may be challenging). Try to deduce a control strategy in the limit $\Delta t \to 0$. Give a plot of $u(x, t)$.

f) Perform a Monte-Carlo simulation of the random walk model for any given set of $u$ at each node. How does the optimal control perform relative to other strategies, such as allocating the resources equally in each asset at each step, or only in the asset with greater expected return $r_1, r_2$, or uniformly at random at each step? How significantly better is the optimal control in maximizing the expected utility? Consider the above questions in both the case with and without transaction costs. In the case of transaction costs can you find a simple control using your financial intuition which gives comparable results to the optimal control? Justify your results using the expressions derived above and Monte-Carlo simulations. Give plots of $\Phi(x, 0)$ for each of the strategies above.

g) How does allowing correlation $\rho \neq 0$ effect the results above?
Project 2: Option Pricing using Stochastic Volatility Models

When using the Black-Scholes-Merton model to price derivative contracts the volatility $\sigma$ of the underlying asset (stock) must be specified. Once this parameter is determined, the price of any contingent claim on the asset can in principle then be determined by applying the Black-Scholes-Merton formula for the given value of volatility $\sigma$. However, in practice, when considering the implied volatility $\sigma$ in the Black-Scholes-Model required to match contract prices realized in the marketplace, it is found that significantly different volatilities are required even when the contracts depend on the same underlying asset. This suggests that the lognormal asset price dynamics assumed in the Black-Scholes-Merton model is insufficient to fully capture the asset price dynamics occurring in the marketplace. In particular, it is found that for call options that the volatility changes for strike prices significantly larger or smaller than the current spot price of the asset, these features are referred to as the volatility smile and volatility skew. One approach to extend the Black-Scholes-Merton theory to capture the volatility smile and skew is to introduce a more sophisticated dynamics for the underlying asset. One natural approach is to allow for the volatility of the underlying asset to evolve according to its own stochastic dynamics.

In this project a stochastic volatility model will be explored for the pricing of contingent claims. An interesting feature of stochastic volatility models is that not only can the volatility skew and smile be described but one can use models calibrated to capture these features in the marketplace to attempt to price the contingent claims based on the realized variance and volatility of assets. In this project the pricing of contingent claims depending on the realized variance and volatility will be considered.

The price of a contingent claim with maturity $T$ under the risk-neutral probability of a stochastic volatility model is given by:

$$ V_0(v) = e^{-rT} E(f(x(T))) $$

$$ = e^{-rT} \int_{-\infty}^{\infty} f(x) \rho(x,v) dx $$

where $\rho(x,v)$ is the probability density (if it exists) and $x = \log(s_t e^{r(T-t)/K})$ is a logarithm of the stock price. For many stochastic volatility models it is often convenient to express the probability in $x$ in terms of a characteristic function at maturity $T$ of the form $\phi_T(k,v)$, which gives:

$$ \rho(x,v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_T(k,v)e^{ikx} dk. $$

To make use of this in practice requires a numerical approximation of the Fourier Transform and the integrals above.

a) In class we discussed the Heston model for stochastic volatility, which has the following characteristic function for the risk-neutral probability:

$$ \phi_T(k,v,t) = \exp(C(k,\tau)\bar{v} + D(k,\tau)v) $$

using the notation in class with $x = \log(s_t e^{r(T-t)/K})$ and $\tau = T - t$. Write a code in matlab to price a general contingent claim depending on both the underlying asset price and stochastic variance under the Heston model. Approximate the integrals representing the expectation and use the Discrete Fast Fourier Transform (DFT) to obtain an approximation of the probability density.
\( \rho(x,v) \) for each fixed \( v \).

b) Use your code from part a) to compute the prices of call and put options with strikes ranging from \( K = \$0.10 : \$0.10 : \$30 \). Some suggested parameters are \( r = 0.05, s_0 = 10, \lambda = 4, \sqrt{v_0} = 30\%, \sqrt{\bar{v}} = 30\%, \eta = 50\%, \rho = 0.5, T = 2 \). Feel free to choose other values, which are possibly more interesting, for this question and the others, but clearly specify them in your report. Give a plot of both the call and put option prices vs strike price.

c) When pricing variance and volatility dependent options in practice, one would like to have a justification for these prices in terms of assets in the marketplace. Ideally one would also like to have a hedging strategy to replicate the value of the option, which mitigates somewhat relying on the specific stochastic model and the parameter specification. The realized variance of an option can in principle be obtained in terms of assets in the marketplace. The fairprice of a variance swap is given by:

\[
E(\sigma^2_R(T)) = \frac{2}{T} \left( rT - \left( \frac{S_0}{\ell} e^{rT} - 1 \right) \right) - \log \left( \frac{\ell}{S_0} \right) + e^{rT} \int_0^\ell \frac{1}{K^2} P(K) dK + e^{rT} \int_\ell^{\infty} \frac{1}{K^2} C(K) dK
\]

where \( P(K), C(K) \) are short-hand for the value of the call and put expiring at maturity \( T \) with strike \( K \). The parameter \( \ell \) is in principle arbitrary and can be chosen for convenience. Compute the fairprice of the variance swap using the code for the Heston model develop in part a), with say 10000 call and put options. Pick a specific value of \( \ell \) and specify it in your report.

d) In approximating the integrals above in practice, only a relatively small number of strike prices \( K \) will be available for options in the market place. Give a plot of the error as the number is decreased for the available strikes for the options. For each number of available strikes use the ones that are spread out as uniformly as possible. About how many strikes appear to be necessary to get good results in practice?

e) Using asymptotic expressions obtained from the optimal path method discussed in class compute the volatility term structure for the implied Black-Scholes-Merton variance \( \sigma^2_{BS}(K,T) \) when the spot price \( S_0 = K \). Give several plots for the different values of \( \rho, \eta \) including the parameters used above. Discuss the observed volatility skew. Can you relate these features at least intuitively to explain some of trends in the option prices you observed.
Project 3: Monte-Carlo Methods for Option Pricing with Asset Jump Diffusion Dynamics

One approach to extending the Black-Scholes-Merton theory for the pricing of options which captures features of the implied volatility smile and skew is to introduce into the stochastic dynamics of the asset prices random jumps. Roughly speaking these jumps can be thought of as a model for unexpected rare world events which upon arrival as news to market investors leads to a rapid change in the asset price over a short period of time. Obtaining useful analytic pricing formulas for such models is often challenging. However, the stochastic process is usually relatively straightforward to directly simulate. This suggests using the Monte-Carlo Method to estimate the option prices. Applying the Monte-Carlo Method in practice to obtain sufficiently accurate numerical estimates of option prices and the “greeks” can be expensive. In this project some basic strategies to reduce the expense of the Monte-Carlo Method when estimating the price of options in a stochastic volatility model with jumps are explored.

A basic extension to the Heston stochastic volatility model which includes jumps is the following:

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu(t)dt + \sqrt{v_t}dB^{(1)}_t + (J - 1)dq_t \\
\frac{dv_t}{v_t} &= -\lambda'(v_t - \bar{v})dt + \eta\sqrt{v_t}dB^{(2)}_t
\end{align*}
\]

where the Brownian motions have correlation \(\langle dB^{(1)}_t dB^{(2)}_t \rangle = \rho dt\) and for simplicity we set \(\mu(t) = r\) with no dividends. The Poisson process \(q_t\) which has jumps in \(S_t\) of fixed proportion \(J\) (i.e. price jumps from \(S_t\) to \(JS_t\)) is formally defined for an interval of time \(dt\) by:

\[
dq_t = \begin{cases} 
1 & \text{with probability } \lambda dt \\
0 & \text{with probability } 1 - \lambda dt 
\end{cases}
\]

where \(\lambda\) is the rate at which the price shocks occur (i.e. \(\frac{1}{\lambda}\) = average time for shock to occur).

A simple discretization of the above SDE’s is to make the change of variable \(Y_t = \log(S_t)\) and use the numerical scheme:

\[
\begin{align*}
Y_{n+1} &= Y_n + \mu_n \Delta t + \sqrt{v_n} \Delta B^{(1)}_n + J' \Delta q_n \\
v_{n+1} &= v_n - \lambda'(v_n - \bar{v}) \Delta t + \eta \sqrt{v_n} \Delta B^{(2)}_n
\end{align*}
\]

where \(\mu_n = r - \frac{1}{2}v_n\), \(J' = \log(J)\). In the scheme the following modification is used whenever \(v_{n+1} < 0\), this is replaced by \(v_{n+1} \rightarrow -v_{n+1}\) (reflecting boundary conditions at \(v = 0\)). The price shocks are modeled over each interval by generating a uniform random variable \(u\) with \(\Delta q_n = 1\) only if \(u < \lambda \Delta t\), and \(\Delta q_n = 0\) otherwise.

a) Write a code in matlab to simulate trajectories of the modified Heston model which incorporates jumps. Compute using the Monte-Carlo method the prices of call options and put options having strikes ranging \(K = 0.5 : 0.5 : 10\). In each case, also estimate the variance \(\sigma^2\) of the random variable \(f(S_T)\) whose mean is being estimated by the Monte-Carlo method. Give a plot of the option prices vs the strike. Also give a plot of the variance \(\sigma^2\) of the estimate vs. the strike for each option. You may choose any parameters you like for the model. Some suggested parameters are \(r = 0.05\), \(s_0 = 10\), \(\lambda' = 4\), \(\lambda = 1\), \(\sqrt{v_0} = 30\%\), \(\sqrt{\bar{v}} = 30\%\), \(\eta = 50\%\), \(\rho = 0.5\), \(T = 2\). Feel free to choose other values, which are possibly more interesting, for this question and the others, but clearly specify
them in your report.

b) Use the method of control variates to reduce the variance of the Monte-Carlo estimates for the call and put options. A natural control variate is to use the Black-Scholes option value at maturity. The control variate variance reduction strategy can be implemented as follows. For each trajectory generated in the modified Heston model, simultaneously compute the standard lognormal stock price dynamics \( S_{t}^{BS} \) for a given volatility, say with \( \sigma_{BS}^2 = \bar{\nu} \) using the same realized Gaussian increments for the Brownian motion as for \( S_{t}^{Heston} \). The random variable \( f(S_{T}^{Heston}) \) can be estimated by the random variable \( f(S_{T}^{BS}) \). Since the value of the option under the Black-Scholes model is known from the Black-Scholes formula, the mean value of \( f(S_{T}^{BS}) \) is known and the method of control variates can be applied. Recompute the call and put option prices from a) using the method of control variates for the Monte-Carlo method. Estimate the variance of the Monte-Carlo estimate for each strike. Give this as a plot of the variance vs strike price. By what factor is this variance smaller? How much computation does this save to achieve a given level of accuracy, say \( \epsilon = 0.1 \)? Try the case both when \( J = 1 \) and \( J = 0.5 \).

c) Obtaining analytic expressions for the "greeks" of an option can be challenging in stochastic volatility and jump diffusion models. One approach is to estimate the "greeks" by using finite difference approximations and computing the corresponding option prices. Compute the \( \Delta \) of a call option using the finite difference approximation:

\[
\Delta \approx \frac{V(S_0 + \Delta s) - V(S_0)}{\Delta s}.
\]

Compute this for the spot price \( S_0 = K, \rho = 0 \), and give a plot of the estimate for \( \Delta s \to 0 \) for \( \Delta s = 0.1 S_0 : 0.3 S_0 : S_0 \) and give a plot of \( \Delta \) vs \( \Delta s \). Also plot the estimate of the Monte-Carlo variance associated with estimating the \( \Delta \) for the different \( \Delta s \).

d) An alternative to using finite differences to approximate the \( \Delta \), is to use Malliavin calculus, which under a few simplifying assumptions, (in particular \( \rho = 0 \)), gives the fairly general class of expressions:

\[
\Delta = \frac{e^{-rT}}{S_0}E \left( f(S_T) \frac{\int_0^T h(s) dW_s}{\int_0^T h(s) \sqrt{\nu_s} ds} \right)
\]

where \( h(s) \) is a deterministic function. For \( h(s) = 1 \) we obtain the expression:

\[
\Delta = \frac{e^{-rT}}{S_0}E \left( f(S_T) \frac{W_T}{\int_0^T \sqrt{\nu_s} ds} \right).
\]

Compute the \( \Delta \) using the Monte-Carlo method with this approach and compare the variance for these estimates with that above. Can you find an \( h \) which performs better than the second estimate (\( h = 1 \))? Try different functions and discuss your findings in the report.