Homework Solutions

1) a)

**Stock Market:**

\[ S_1 = 61.77 \]
\[ S_2 = 50.06 \]
\[ S_3 = 40.54 \]
\[ S_4 = 32.88 \]
\[ S_5 = 76.81 \]

**European Option Value (put):**

\[ V_0 = 4.065 \]
\[ V_1 = 16.342 \]
\[ V_2 = 0 \]
\[ V_3 = 17.12 \]
\[ V_4 = 0 \]
\[ V_5 = 0 \]

**Replicating Portfolio:**

Node 1:

\[ w_1 = -0.9856 \text{ (stock)} \]
\[ w_2 = 49.53 \text{ (bond)} \]

Node 2:

\[ w_1 = 0 \text{ (stock)} \]
\[ w_2 = 0 \text{ (bond)} \]

**Risk-Neutral Probabilities:**

Node 1:

\[ q = 0.5004 \text{ (up)} \]

Node 2:

\[ q = 0.5004 \text{ (up)} \]

Node 0:

\[ q = 0.5004 \text{ (up)} \]

\[ V_0 = e^{-2r\Delta t} \left[ f_5 q^5 + 2f_4 (1-q) q + f_3 (-q)^3 \right] \]

\[ = 4.065 \]
b) **American Option Value (put):**

\[ V_1 = 4.6095 \]

\[ V_3 = 17.12 \]

early exercise \( (K - S_0) > V_1^{\text{European}} \)

The **American option (put)** is worth more:

\[ V_0^{\text{American}} - V_0^{\text{European}} = 4.6095 - 4.065 = 0.5445 \]

c) \( D = 7\% \) \( S_0 \)

In this case, we hedge the option using the stock and reinvested dividend yield. The replicating portfolio over a period becomes:

\[ w_1 \sup e^{D\Delta t} + w_2 = f_{\uparrow} \]

\[ w_1 \downarrow down e^{D\Delta t} + w_2 = f_{\downarrow} \]

\((S^* e^{D\Delta t} \text{ is the value of the stock + dividend to the owner of the stock})\)

\[ w_1 = \frac{f_{\uparrow} - f_{\downarrow}}{e^{D\Delta t}sup - e^{D\Delta t}down} \]

\[ w_2 = \frac{e^{D\Delta t}sup f_{\downarrow} - e^{D\Delta t}down f_{\uparrow}}{e^{D\Delta t}sup - e^{D\Delta t}down} \]

This implies...
1. \[ q = \frac{e^{r \Delta t} - d^{e^{r \Delta t}}}{u^{e^{r \Delta t}} - d^{e^{r \Delta t}}} = \frac{e^{(r-D)\Delta t} - d}{u - d} \]

2. \[ f_{new} = w_1 f_{now} + w_2 e^{-r \Delta t} = e^{-r \Delta t} [q f_{up} + (1-q) f_{down}] \]

**European Put:**

- \[ V_2 = 0 \]
- \[ V_5 = 0 \]
- \[ V_3 = 19.35 \]
- \[ V_4 = 0 \]
- \[ V_1 = 19.429 \]

**American Put:**

- \[ V_3 = 0 \]
- \[ V_5 = 0 \]
- \[ V_4 = 0 \]
- \[ V_1 = 10.85 \]
- \[ V_3 = 19.35 \]

There are two ways to construct the model for the underlying stock process:

1. \[ M = r - \frac{1}{2} \sigma^2 \] for \( N \to \infty \) will give \( q \to \frac{1}{2} \) and correct valuation.
2. \( M = r - D - \frac{1}{2} \sigma^2 \) for \( N \to \infty \) will give \( q \to \frac{1}{2} \) and correct valuation.

Method (ii) will give a more accurate result for \( N \to \infty \), so is preferred, while both are technically correct.

We price using \( M = r - D - \frac{1}{2} \sigma \) \( (q = 5.004) \)

The put options are now more valuable in the presence of dividends as expected, since the stock values are depressed by the dividend payments.
Replicating Portfolio over first period:

\[
\begin{align*}
    w_1 (S_{up} + p) + w_2 &= f_{up} \\
    w_1 (S_{down} + p) + w_2 &= f_{down}
\end{align*}
\]

\[
w_1^* = \frac{f_{up} - f_{down}}{S_{up} - S_{down}}
\]

\[
w_2^* = \frac{(S_{up} + p)f_{down} - (S_{down} + p)f_{up}}{S_{up} - S_{down}}
\]

\[f_{now} = w_1^* S_{now} + w_2^* e^{-r \Delta t} = e^{-r \Delta t} \left[ q f_{up} + (1-q) f_{down} \right] \]

\[
=> q = \frac{e^{-r \Delta t} - (S_{down} + p)}{S_{up} - S_{down}}
\]

European Put:
\[
\begin{align*}
    v_0 &= 5.44 \\
    v_1 &= 8.34 \\
    v_2 &= 17.12 \\
    v_3 &= 0 \\
    v_4 &= 0
\end{align*}
\]

American Put:
\[
\begin{align*}
    v_0 &= 6.17 \\
    v_1 &= 9.456 \text{ (Early exercise)} \\
    v_2 &= 17.12
\end{align*}
\]
2d) \( P(\sigma) \) is maximum around the strike \( K \).

e) More valuable for both calls and puts since this increases the probability of being in-the-money. (Indicated by Vega > 0)

Note: being deep in the money compensates for increased risk of being out-of-the-money.
(b) Put option increases in value as the dividend increases, since dividends depress the value of the stock. Risk-neutral stock price dynamics has drift $(r - D - \frac{1}{2} \sigma^2)$.

Call options decrease in value since the dividend depresses the stock price leading to a smaller probability of being in-the-money.
4) (i) \( X_t = B_t \)
\[
dX_t = \sqrt{d} dt + 2B_t dB_t
\]
(ii) \( X_t = 1 + t + e^{B_t} \)
\[
dX_t = (1 + \frac{1}{2} e^{B_t}) dt + e^{B_t} dB_t
\]
(iii) \( X_t = B_t^\alpha(t) + B_t^\gamma(t) \), \( B_t \) indep.
\[
dX_t = 2 dt + 2 B_t^{\alpha''} dB_t^{\alpha'''} + 2 B_t^{\gamma''} dB_t^{\gamma'''}
\]
(iv) \( X_t = \cos(B_t) \)
\[
dX_t = -\cos(B_t) dt - \sin(B_t) dB_t
\]

b) Show directly
\[
\int_0^t B_s^3 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s^3 ds
\]
\[
\frac{1}{3} B_t^3 = \sum_{k=0}^{n} \frac{1}{3} (B_{k+1}^3 - B_k^3) = \frac{1}{3} B_{1/3}^3
\]
\[
= \sum_{k=0}^{n} \frac{1}{3} (B_{k+1}^3 - B_k^3) + \frac{1}{3} B_k^3 (B_{k+1}^3 - B_k^3)
\]
\[
+ \frac{1}{3} B_k^3 (B_{k+1}^3 - B_k^3)
\]
\[
= \sum_{k=0}^{n} B_k (B_{k+1}^3 - B_k^3)
\]
\begin{align*}
I_1^{(n)} &\to S_0^t B_s \, ds \\
I_2^{(n)} &\to S_0^t B_s^2 \, dB_s \\
I_3^{(n)} &\to 0 \\
\text{Ito's Lemma:} \\
Y_t &= f(t, X_t), \quad dX_t = \alpha(t, B_t) \, dt + b(t, B_t) \, dB_t \\
1Y_t &= \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial X} \, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} \, dX_t^2 \\
dt \cdot dt = 0 = dt \cdot dB_t, \quad dB_t \cdot dB_t = dt \ (formally) \\
\text{Let } X_t = B_t, \ Y_t = \frac{1}{2} B_t^2, \ \text{then} \\
1(\frac{1}{2} B_t^2) &= \frac{1}{2} (3 \cdot B_t^2) \, dX_t + \frac{1}{2} (\frac{1}{2} \cdot 3 \cdot 2 B_t) \, dB_t^2 \\
&= B_t \, dt + B_t^2 \, dB_t \\
\Rightarrow \frac{1}{2} B_t^2 &= \int_0^t B_s \, ds + \int_0^t B_s^2 \, dB_s. \\
1) \ X_t &= e^{\alpha t + \alpha B_t} = f(t, B_t) \\
dX_t &= \frac{\partial f}{\partial t} \, dt + \frac{\partial f}{\partial B} \, dB_t + \frac{1}{2} \frac{\partial^2 f}{\partial B^2} \, dB_t^2 \\
&= cX_t \, dt + \alpha X_t \, dB_t + \frac{1}{2} \alpha^2 X_t \, dt \\
&= \left(1 + \frac{1}{2}\right) X_t \, dt + \alpha X_t \, dB_t.
\end{align*}
e) (i) \[ Z = S_0^1 d B_s \], \quad \text{Var}[Z] = E[Z^2] - E[Z]^2 \\
E[Z] = 0 \\
E[Z^2] = E\left[ (S_0^1 d B_s)^2 \right] = S_0^1 s \, ds = \frac{1}{3} \\
\text{Ito's Isometry} \\
(ii) \[ Z = S_0^3 e^{s^2} d s + S_0 e^{s^3} d B_s \] \\
E[Z] = S_0^3 e^{s^2} \, ds \\
E[Z^2] = E\left[ (S_0^3 e^{s^2} d B_s)^2 \right] = S_0^3 e^{2s} \, ds = \left[ \frac{1}{2} e^{2s} \right]_0^1 \\
(iii) \[ Z = \int_0^1 \cos(s) d s + \int_0^1 e^{s^3} s \, d B_s \] \\
E[Z] = \int_0^1 \cos(s) d s \\
E[Z^2] = E\left[ (\int_0^1 e^{s^3} s \, d B_s)^2 \right] = \int_0^1 e^{2s^3} s^2 \, ds \\
= \left[ \frac{1}{4} e^{2s^3} \right]_0^1 \\
(iv) \[ Z = \int_0^1 B_s^2 d B_s \] \\
E[Z] = E\left[ \int_0^1 B_s^2 d B_s \right] = E\left[ -S_1^\gamma \frac{1}{2} \, ds + \frac{1}{2} B_t^2 \right] \\
= \frac{1}{2} (t-1) \\
E[Z^2] = E\left[ (\int_0^1 B_s^2 d B_s)^2 \right] = E\left[ \int_0^1 B_s^2 \, ds \right] \\
= S_1^\gamma s \, ds = \left[ \frac{1}{2} s \right]_1^2
(This question has slight technical problem since $\sqrt{\cos(s)} \notin \mathbb{C}$ for $\cos(s) < 0$)