Instructions: Please be sure to answer each of the questions carefully and to show all of your work for maximum credit. If you have any questions please feel free to ask.

Problem 1: Markowitz Portfolio Theory
Consider three assets with returns $r_1$, $r_2$, and $r_3$. Assume that the mean returns are

\[ \mu = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \]

(1)

with covariances given by

\[ V = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \]

(2)

Further, you may use the following:

\[ V^{-1} = \begin{bmatrix} 6 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \]

(3)

and $A = 1^T V^{-1} \mu = 35$, $B = \mu^T V^{-1} \mu = 59$, $C = 1^T V^{-1} 1 = 21$, and $D = 14$.

Answer all of the following questions under the assumptions of Markowitz's Portfolio Theory.
a) In what proportions \(\{w_i\}_{i=1}^3\) would you invest in each of the assets to achieve a mean return of 4?

\[
W_p = g + \frac{1}{D} \left( B V^{-1} 1 - A V^{-1} \mu \right) = \frac{1}{14} \begin{bmatrix}
1.9 \\
-4
\end{bmatrix}
\]

\[
h = \frac{1}{D} \left( CV^{-1} \mu - AV^{-1} 1 \right) = \frac{1}{14} \begin{bmatrix}
-2.3 \\
0.7
\end{bmatrix}
\]

\[
W_p = \frac{1}{14} \begin{bmatrix}
1.9 \\
-4
\end{bmatrix} + \frac{4}{14} \begin{bmatrix}
-7 \\
0
\end{bmatrix} = \frac{1}{14} \begin{bmatrix}
-9 \\
4
\end{bmatrix} \approx \begin{bmatrix}
-0.6429 \\
0.2857
\end{bmatrix}
\]

\[
\begin{bmatrix}
1.3871
\end{bmatrix}
\]

b) What is the variance of this return?

\[
\sigma_p^2 = W_p^T V W_p = \frac{1}{D} \left( \mu_p - \frac{A}{c} \right)^2 + \frac{1}{C}
\]

\[
= \frac{1}{Dc} \left( \mu_c - A \right)^2 + \frac{D}{Dc}
\]

\[
= \frac{49^2 + 14}{294} = \frac{2415}{294} \approx 8.2143
\]
c) In what proportions \( \{w_i\}_{i=1}^3 \) would you invest in each of the assets to achieve a mean return of 1?

\[
W_p = g + M_p h
\]

\[
= \frac{1}{14} \left[ \begin{array}{c} 14 \\ -9 \end{array} \right] + \frac{1}{14} \left[ \begin{array}{c} -7 \\ 7 \end{array} \right] = \frac{1}{14} \left[ \begin{array}{c} 7 \\ 7 \end{array} \right] = \frac{1}{7} \left[ \begin{array}{c} 6 \\ -1 \end{array} \right]
\]

\[
\approx \left[ \begin{array}{c} 0.8521 \\ 0.2857 \end{array} \right]
\]

\[
\approx \left[ \begin{array}{c} 0.1479 \end{array} \right]
\]

d) Is this portfolio efficient?

\[
M_p = 1, \quad \frac{A}{C} = \frac{35}{21} = 1.6666 > M_p \quad \Rightarrow \quad M_p < \frac{A}{C}
\]

the portfolio is inefficient

(the min. variance portfolio has return \( \frac{A}{C} > M_p \))
e) If a risk-free asset is introduced having mean return 0.5, in what proportions \( \{w_i\}_{i=1}^3 \) do you invest to achieve a mean return of 1?

\[
W_{p+} = \left( \frac{M_{p+} - r_0}{H} \right) V^{-1} \left( \mu - r_0 \mathbf{1} \right), \quad H = C r_0 - 2A r_0 + B
\]

\[
= \frac{117}{8} \times \frac{1}{2} \begin{bmatrix} 25 \\ 14 \\ 10 \end{bmatrix} = \frac{117}{16} \begin{bmatrix} 25 \\ 14 \\ 10 \end{bmatrix}
\]

\[
= \frac{21}{4} - 35 + 59 = \frac{117}{4}
\]

\[
V^{-1} \left( \mu - r_0 \mathbf{1} \right) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 25 \\ 14 \\ 10 \end{bmatrix}
\]

\[
\approx \begin{bmatrix} 12.5 \\ 7 \end{bmatrix}
\]

f) Is this portfolio efficient? Yes, since portfolio \( M_{p+} > r_0 \).

(While \( M_{p+} < A/c \), when we include the risk-free asset with weight \( w_0 = 1 - \sum_{i=1}^3 w_i \), the variance is much less than the portfolio in part c, so there is no better portfolio with larger return for the same level of risk. The relevant criterion for the risk-free asset included is \( M_{p+} > r_0 \).)
Problem 2: Stock Market Models

Figure 1: Binomial Market Model

a) Does the above market model with $s_0 = 100, s_1 = 104, s_2 = 110$ admit an arbitrage opportunity when the interest rate $r$ is such that $e^{r\Delta t} = 1.05$? If so, give a trading strategy which exploits the arbitrage?

The no-arbitrage condition is: $S_{\text{down}} < e^{r\Delta t} S_{\text{now}} < S_{\text{up}}$

$e^{r\Delta t} S_{\text{now}} = 1.05 \cdot 100 = 105$, $S_{\text{down}} = 104$, $S_{\text{up}} = 110$, hence no arbitrage.

b) Does the above market model with $s_0 = 100, s_1 = 106, s_2 = 110$ admit an arbitrage opportunity when the interest rate $r$ is such that $e^{r\Delta t} = 1.05$? If so, give a trading strategy which exploits the arbitrage?

$e^{r\Delta t} S_{\text{now}} = 1.05 \cdot 106 = 110$, $S_{\text{up}} = 110$, so

$S_{\text{down}} < e^{r\Delta t} S_{\text{now}} < S_{\text{up}}$ arbitrages opportunity exists. In particular, make a loan of 100 from the bank, buy 1 unit stock. This yields $P_{\text{up}} = S_{\text{up}} - e^{r\Delta t} S_{\text{now}} = 110 - 105 = 5$

$P_{\text{down}} = S_{\text{down}} - e^{r\Delta t} = 106 - 105 = 1$

Figure 2: Incomplete Trinomial Market Model

(c) Use the principle of no arbitrage to obtain bounds on the value a call option with strike $K = 100$ written on the above incomplete trinomial stock market when $s_0 = 100, s_1 = 90, s_2 = 100, s_3 = 110$ and $r = 0$. (Hint: To obtain the upper bound compute the replicating portfolio consisting of the
Problem 2: Binomial Stock Market Model

Consider a market model for a stock over two years which has \( \exp(r\Delta t) = \frac{3}{2} \) and prices \( s_0 = 40, s_1 = 20, s_2 = 100, s_3 = 10, s_4 = 40, s_5 = 370 \).

a) Compute the risk-neutral probability of an upward price movement of the stock at each node, label them \( q_0, q_1, q_2 \).

\[
q = \frac{e^{r\Delta t} s_{\text{up}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}} \Rightarrow q_0 = \frac{\frac{3}{2} \cdot 40 - 20}{100 - 20} = \frac{60 - 20}{80} = \frac{1}{2}.
\]

\[
q_1 = \frac{\frac{3}{2} \cdot 20 - 10}{40 - 10} = \frac{20}{3}.
\]

\[
q_2 = \frac{\frac{3}{2} \cdot 100 - 40}{50 - 40} = \frac{140}{5} = 28.
\]

b) Compute the value of a European put option with strike price \( K = 46 \).

\[
V_0 = e^{-r\Delta t} E_{N} [ f(S_T) ] = (q_0 \cdot q_2) \cdot f_5 + (1-q_0) \cdot q_1 \cdot f_4 + (1-q_0) \cdot q_1 \cdot f_4 + (1-q_0) \cdot q_2 \cdot f_3.
\]

\[
= \left( \frac{1}{2} \right)^2 \left[ \frac{1}{6} \cdot 0 + \frac{1}{2} \cdot \frac{1}{3} \cdot 6 + \frac{1}{2} \cdot \frac{1}{3} \cdot 6 + \frac{1}{2} \cdot \left( \frac{1}{3} \right) \cdot 36 \right]
\]

\[
= \left( \frac{1}{2} \right)^2 \left[ 2 + 2 + 2 \right] = \frac{40}{9} \approx 4.4.
\]

c) Compute the value of an American put option with strike price \( K = 46 \). Are they the same or different?

Here we must work inductively:

\[
f_5 = e^{-r\Delta t} \left[ q_2 \cdot f_5 + (1-q_2) \cdot f_4 \right]
\]

\[
= \left( \frac{3}{2} \right) \left[ \frac{1}{2} \cdot 10 + \left( \frac{1}{2} \right) \cdot 1 \right] = \frac{21}{2}.
\]

\[
f_4 = e^{-r\Delta t} \left[ q_1 \cdot f_4 + (1-q_1) \cdot f_3 \right]
\]

\[
= \left( \frac{3}{2} \right) \left[ \frac{1}{3} \cdot 10 + \left( \frac{1}{3} \right) \cdot 1 \right] = \frac{23}{3}.
\]

\[
f_3 = \max \left( f(s_1), f_1 \right) = \max \left( 26, \frac{23}{3} \right) = 26.
\]

\[
f_0 = \left( \frac{3}{2} \right) \left[ q_0 \cdot f_0 + (1-q_0) \cdot f_1 \right] = \left( \frac{3}{2} \right) \left[ \frac{1}{2} \cdot \frac{8}{3} + \frac{1}{2} \cdot 26 \right] = \frac{1}{8} + \frac{36}{3} = \frac{33}{8}.
\]

\( \approx 9.79 \)
Problem 2: Black-Scholes-Merton Formula

The Black-Scholes-Merton Formula for a call and put option with continuous dividend yield $D$ is:

\[ c(s_0, K, T) = s_0 e^{-DT} N(d_1) - Ke^{-rT} N(d_2) \]

\[ p(s_0, K, T) = Ke^{-rT} N(-d_2) - s_0 e^{-DT} N(-d_1) \]

with

\[ d_1 = \frac{1}{\sqrt{T}} \left( \log \left( \frac{s_0 e^{-DT}}{K} \right) + (r + \frac{1}{2} \sigma^2)T \right) \]

\[ d_2 = \frac{1}{\sqrt{T}} \left( \log \left( \frac{s_0 e^{-DT}}{K} \right) + (r - \frac{1}{2} \sigma^2)T \right) \]

where $d_2 = d_1 - \sigma \sqrt{T}$ and

\[ N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy \]

denotes the cumulative distribution function of the standard normal distribution.

a) Give an expression for the sensitivities $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ of a call option when the dividend $D$ is varied? What relation does this have to the option’s Delta and Gamma? (Hint: Use the chain rule with $y = s_0 e^{-DT}$.)

Let $y = s_0 e^{-DT}$, then

\[ \Delta_0 = \frac{\partial V}{\partial S_0} = \frac{\partial V}{\partial y} \frac{\partial y}{\partial S_0} = \Delta_0 e^{-DT} \]

where $\Delta_0$ is the Delta of the non-dividend paying option at spot $y$. Let $\Gamma_0 = \frac{\partial^2 V}{\partial S_0^2} = \frac{\partial \Delta_0}{\partial y} \frac{\partial y}{\partial S_0} = \Gamma_0 e^{-DT}$.

\[ \frac{\partial V}{\partial D} = \frac{\partial V}{\partial y} \frac{\partial y}{\partial D} = \Delta_0 s_0 e^{-DT} \cdot (-T) = -\Delta_0 s_0 T e^{-DT} = -\Delta_0 s_0 T = -\Delta_D s_0 T. \]

\[ \frac{\partial^2 V}{\partial D^2} = \frac{\partial}{\partial D} \left( -\Delta_0 s_0 T e^{-DT} \right) = -\frac{\partial \Delta_0}{\partial D} s_0 T e^{-DT} - \Delta_0 s_0 T \frac{\partial}{\partial y} \left( -T \right) e^{-D} \]

\[ = -\frac{\partial \Delta_0}{\partial y} \left( \frac{\partial y}{\partial D} \right) s_0 T e^{-DT} + \Delta_0 s_0 T \frac{\partial}{\partial D} e^{-D} \]

\[ = \Gamma_0 s_0^2 T^2 e^{-DT} + \Delta_0 s_0 T \frac{\partial}{\partial D} e^{-D} = s_0 T \Gamma_0 + s_0 T \Delta_D. \]

b) What happens to the value of a call option when the dividend is increased?

The value decreases, since $\frac{\partial V}{\partial D} < 0$. (Use $\Delta_0 = N(d_1) > 0$)

c) Give the corresponding expression for the put option sensitivities $\frac{\partial V}{\partial D}$ and $\frac{\partial^2 V}{\partial D^2}$.

Same as the above with $\Delta_0 = -N(-d_1)$.

\[ \Gamma_0 = \frac{1}{\sqrt{2\pi} \sigma \sqrt{T}} e^{-\frac{d_1^2}{2}}. \]
d) What happens to the value of a put option when the dividend is increased?

The put option increases in value since \( \frac{\partial V}{\partial D} < 0 \), using \( \Delta D < 0 \).