Homework 1

Due date: Sep 23

1. For the equations \( x + y = 4 \), \( 2x - 2y = 4 \), draw the row picture (Two intersecting lines) and the column picture (combination of two columns equal to the column vector \((4, 4)\) on the right hand side) (Problem set 1.2, exercise 3)

   Solution:

   ![Row and Column Pictures](image.png)

2. Describe the intersection of the three planes \( u + v + w + z = 6 \) and \( u + w + z = 4 \) and \( u + w = 2 \) (all in four dimensional space). Is it a line or a point or an empty set? What is the intersection if the fourth plane \( u = -1 \) is included? Find a fourth equation that leaves us with no solution. (Problem set 1.2, exercise 5)

   Solution: It is a line, given by

   \[
   (u, v, w, z) = (t, 2, 2 - t, 2)
   \]

   b) The solution with \( u = -1 \) is a point given by \((-1, 2, 3, 2)\)
c) There is no solution with \( v = -3 \) for example. Any choice for \( v = c \) and \( z = c \) works as long as \( c \neq 2 \).

3. Consider the column picture corresponding to the following singular system

\[
\begin{bmatrix}
1 \\
1
\end{bmatrix} + \begin{bmatrix}
2 \\
1
\end{bmatrix} + w \begin{bmatrix}
3 \\
2
\end{bmatrix} = \bar{b}.
\]

Show that the three columns on the left lie in the same plane by expressing the third column as a combination of the first two. What are all the solutions \((u, v, w)\) if \(\bar{b} = (0, 0, 0)\). (Problem set 1.2, exercise 11)

**Solution:** \((-c, 2c, -c)\)

4. If \((a, b)\) is a multiple of \((c, d)\) with \(abcd \neq 0\), show that \((a, c)\) is a multiple of \((b, d)\). You could use numbers first to see how \(a, b, c\) and \(d\) are related. The question corresponds to the following deep fact, if

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

has linearly dependent rows, then it has dependent columns. (Problem set 1.2, exercise 21)

**Solution:** \((a, b) = k (c, d) \implies a = kc, b = kd.\)

Thus, \((a, c) = (kc, c) = c(k, 1).\)

Thus \((b, d) = (kd, d) = d(k, 1).\)

Thus \((a, c) = c/d (b, d).\)

5. Reduce this system to upper triangular form by two row operations:

\[
\begin{align*}
2x + 3y + z &= 8 \\
4x + 7y + 5z &= 20 \\
-2y + 2z &= 0.
\end{align*}
\]

Circle the pivots. Solve by back-substitution for \(z, y, x\). (Problem set 1.3, exercise 12)

**Solution:**

\[
\begin{bmatrix}
2 & 3 & 1 & 8 \\
4 & 7 & 5 & 20 \\
0 & -2 & 2 & 0
\end{bmatrix}
\]

\(\rightarrow \begin{bmatrix}
2 & 3 & 1 & 8 \\
0 & 1 & 3 & 4 \\
0 & -2 & 2 & 0
\end{bmatrix}
\)

\(\rightarrow \begin{bmatrix}
2 & 3 & 1 & 8 \\
0 & 1 & 3 & 4 \\
0 & 0 & 8 & 8
\end{bmatrix}
\)
Using back substitution we then get \( z = 1, y = 1 \) and \( x = 2 \).

6. It is impossible for a system of linear equations to have exactly two solutions. Explain why. a) If \((x, y, z)\) and \((X, Y, Z)\) are two solutions, what is another one? b) If 25 planes meet at two points, where else do they meet? (Problem set 1.3, exercise 15)

**Solution:** a) \( \left( \frac{x+X}{2}, \frac{y+Y}{2}, \frac{z+Z}{2} \right) \)
b) In the line passing through those two points.

7. Consider the following system of equations

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

True or false:
a) If \(a_{31} = 0\), then no multiple of equation 1 will be subtracted from equation 3? **True**
b) If \(a_{32} = 0\), then no multiple of equation 2 will be subtracted from equation 3? (Answer this part independent of part a) **False**
c) If the third equation has \(a_{31} = a_{32} = 0\), then no multiple of equation 2 will be subtracted from equation 3? (Problem set 1.3, exercise 26) **True**

8. If the entry in the \( i \)th row and the \( j \)th column of \( A \) is \( a_{ij} \), use this subscript notation to write a) the first pivot, b) the multiplier \( l_{i1} \) of row 1 to eliminate the first variable from row \( i \), c) the new entry that replaces \( a_{ij} \) for \( j = 1, 2, 3 \ldots n \), after that subtraction, d) the second pivot. (Problem set 1.4, exercise 14)

**Solution:** a) \( a_{11} \)
b) \( \frac{a_{1j}}{a_{11}} \)
c) \( a_{ij}^{\text{new}} = a_{ij}^{\text{old}} - \frac{a_{1j}^{\text{old}}}{a_{11}^{\text{old}}} a_{1j}^{\text{old}} \)
d) \( a_{22}^{\text{new}} = a_{22}^{\text{old}} - \frac{a_{21}^{\text{old}}}{a_{11}^{\text{old}}} a_{12}^{\text{old}} \)

9. Describe the rows of \( EA \) and the columns of \( AE \) if

\[
E = \begin{bmatrix}
1 & 7 \\
0 & 1
\end{bmatrix}
\]
Solution: a) The first row of $EA$ is row 1 of $A + 7$ times row 2 of $A$. The second row of $EA$ is the second row of $A$

b) The first column of $AE$ is column 1 of $A$. The second column of $AE$ is 7 times column 1 of $A +$ column 2 of $A$. 