Theorem 1. Let \( \{a_n\} \) be a sequence of complex numbers such that
- \( 0 < |a_1| \leq \ldots \)
- \( \lim_{n \to \infty} |a_n|^{-1} = 0. \)

Then there exists an entire function \( f : \mathbb{C} \to \mathbb{C} \) whose zeroes coincide with \( \{a_n\} \).

Theorem 2. Every entire function \( f : \mathbb{C} \to \mathbb{C} \) has a representation
\[
f(s) = e^{h(x)s} \prod_{n=1}^{\infty} (1 - \frac{s}{a_n})e^{\sum_{j=1}^{n-1} \frac{1}{j} \left(\frac{s}{a_n}\right)^j},
\]
where \( h(s) \) is an entire function.

Theorem 3. Let \( f \) be entire, \( f(0) \neq 0 \), with \( \alpha(f) < \infty \). Then
- \( \beta \leq \alpha \),
- \( h(s) \) is a polynomial of degree \( \deg(h) \leq \alpha \),
- \( \alpha = \max(\beta, \deg(h)) \).

Gamma function

Let \( \gamma \) be Euler’s constant and define
\[
\frac{1}{\Gamma(s)} := se^{-\gamma s} \prod_{n \geq 1} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}}.
\]

Gamma function satisfies the following properties:
- \( \Gamma(s) = \frac{1}{s} \prod_{n \geq 1} \left(1 + \frac{1}{n}\right)^s \left(1 + \frac{s}{n}\right)^{-1} \).
\* \* \* \\
• \[\Gamma(s) = \lim_{n \to \infty} \frac{1 \cdot 2 \cdots (n - 1) \cdot n}{s \cdot (s + 1) \cdots (s + n - 1)},\]
• \[\Gamma(s + 1) = s \Gamma(s),\]
• for \(s \in \mathbb{C} \setminus \mathbb{Z},\)
\[\Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)},\]
• \[\Gamma(1/2) = \sqrt{\pi},\]
• \[\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt,\]
• for \(\arg(s) \in [-\pi + \delta, \pi - \delta]\)
\[\log(\Gamma(s)) = (s - 1/2) \log(s) - s + \log(\sqrt{2\pi}) + O(|s|^{-1}).\]

Applications to \(\zeta(s)\)

**Theorem 4.** The Riemann zeta function has the following properties:
• if \(\rho\) is a nontrivial zero \(\rho\) of \(\zeta(s)\) then \(\Re(\rho) \in [0, 1],\)
• there are infinitely many nontrivial zeroes,
• 
\[\sum \frac{1}{|\rho_n|} = \infty\]
(over all nontrivial zeroes),
• for all \(\epsilon > 0\) one has
\[\sum \frac{1}{|\rho_n|^{1+\epsilon}} < \infty.\]

**Theorem 5** (Vallée-Poussin). For all \(\epsilon > 0\) there is a constant \(c > 0\) such that \(\zeta(s) \neq 0\), provided
\[\Re(s) \geq 1 - \frac{c}{\log(|\Im(s)| + 2)}.\]