Exercises 1 (January 23, 2006)

1. According to Chevalley-Warning, a hypersurface $X_f \subset \mathbb{P}^n$ over a finite field has a rational point, provided the degree of the defining form $\deg(f) \leq n$. Find
   
   • a smooth cubic curve $C \subset \mathbb{P}^2$, over $\mathbb{F}_7$, with $C(\mathbb{F}_7) = \emptyset$,
   • a smooth quartic surface $S \subset \mathbb{P}^3$, over $\mathbb{F}_5$, with $S(\mathbb{F}_5) = \emptyset$.

2. Let $k$ be an infinite field and $f \in k[x_1, \ldots, x_n]$ a nonzero polynomial. Show that there exists an $a = (a_1, \ldots, a_n) \in \mathbb{A}^n(k)$ such that $f(a) \neq 0$.

3. Show that not every ideal in $\mathbb{C}[x, y]$ is principal.

4. Let $X \subset \mathbb{A}^2$ be a curve given by $y^2 = x^3$. Show that every element of the ring $k[X]$ can be uniquely written in the form $f(x) + g(x)y$, with $f, g \in k[x]$.

5. Consider the map $\phi : \mathbb{A}^3 \rightarrow \mathbb{A}^3$ given by
   
   $\phi(x, y, z) := (x, xy, xyz)$.

   Determine the image $\phi(\mathbb{A}^3)$. Is this set open in $\mathbb{A}^3$?