Exercise #10

Proposed Solution
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1. Any single clause c in a DNF formula is a 1-CNF formula, so we can polynomially check its satisfiability (i.e., verify that if x appears in c, $\neg x$ doesn't appear and vice versa).

   Now, the whole DNF formula ϕ is satisfiable if and only if *any* of its clauses is satisfiable, so we can consider the clauses one by one and return $\bigvee_{c\in\phi}$ 1-CNF-$\textsc{Sat}(c)$.

2. (a) Assuming $NP \neq \text{co-}NP$, $\exists L \in NP \backslash \text{co-}NP$[1] (e.g., any NP-complete problem). Consider $\Sigma^* \in P \subseteq \text{co-}NP$. Now, $L = L \cap \Sigma^* \in DP$ but $L \notin \text{co-}NP \supseteq NP \cap \text{co-}NP$.

   (b) Let $L \in DP$. Then $L = L_1 \cap L_2$ where $L_1 \in NP$ and $L_2 \in \text{co-}NP$. Let M be a polynomial NDTM accepting L_1 and let O be an oracle for $\overline{L_2} \in NP$.

   Consider a NDTM M' which simulates M, queries O and accepts if and only if M accepted and the O rejected. It's easy to see that M accepts L and $M' \in NP^{NP} = \Sigma_2$.

   (c) Consider $L_1 = 3\text{-}\textsc{Sat} \times 3\text{-CNF}$ and $L_2 = 3\text{-CNF} \times \overline{3\text{-}\textsc{Sat}}$. As $3\text{-CNF} \in P$, we have $L_1 \in NP, L_2 \in \text{co-}NP$ so $\textsc{Sat}\text{-}\textsc{UnSat} = 3\text{-}\textsc{Sat} \times \overline{3\text{-}\textsc{Sat}} = L_1 \cap L_2 \in DP$.

   (d) Let $L \in DP$. Then $L = L_1 \cap L_2$ where $L_1 \in NP$ and $L_2 \in \text{co-}NP$. Since 3-$\textsc{Sat}$ is NP-complete, we have a polynomial reduction f from L_1 to 3-$\textsc{Sat}$; since $\overline{3\text{-}\textsc{Sat}}$ is co-NP-complete, we have a polynomial reduction g from L_2 to $\overline{3\text{-}\textsc{Sat}}$.

   Consider the reduction $h(x) = (f(x), g(x))$. We have

   $$x \in L \Leftrightarrow x \in L_1 \wedge x \in L_2 \Leftrightarrow f(x) \in 3\text{-}\textsc{Sat} \wedge g(x) \in \overline{3\text{-}\textsc{Sat}} \Leftrightarrow h(x) \in \textsc{Sat}\text{-}\textsc{UnSat}$$

   and h is polynomial as f, g are.

3. $\textsc{Sat} \in NP \subset EXP$, so there exists a deterministic TM M accepting $\textsc{Sat}$. Consider the deterministic TM M' which simulates M and halts if and only if M accepts.

   Let $\langle M' \rangle$ be a description of M', and consider the reduction $f(\phi) = (\langle M' \rangle, \phi)$ from SAT to $\textsc{Halt}$.

   Correctness: $\phi \in \textsc{Sat} \Leftrightarrow M$ accepts $\phi \Leftrightarrow M'$ halts on ϕ.

   Space Complexity: Logarithmic, as $\langle M' \rangle$ is constant and ϕ should be simply copied as-is.

4. We prove this by induction over t.

   For the base case $t = 0$, by $\textsc{Reach}$'s definition $\textsc{Reach}(C_1, C_2, 1) = \mathit{true} \Leftrightarrow \delta(C_1) = C_2$ which is equivalent to the existence of a path of length $\leq 2^0 = 1$ in the graph[2].

   Assuming correctness for t, we consider $t + 1$. $\textsc{Reach}(C_1, C_2, t + 1) = \mathit{true}$ is equivalent to $\exists C_{mid} \quad (\textsc{Reach}(C_1, C_{mid}, t) = \mathit{true}) \wedge (\textsc{Reach}(C_{mid}, C_2, t) = \mathit{true})$.

   By the induction hypothesis, this is equivalent to the existence of two paths $C_1 \rightsquigarrow C_{mid}$ and $C_{mid} \rightsquigarrow C_2$ of length $\leq 2^t$ each; this is equivalent[3] to a path $C_1 \rightsquigarrow C_2$ of length $\leq 2^{t+1}$.

[1] Why is $NP \subset \text{co-}NP$ impossible?
[2] We assume $C_1 \neq C_2$, of course.
[3] ($\Rightarrow$) concatenate the two paths; ($\Leftarrow$) split the path in half.