1. We show that 2-SAT is co-NL-hard by reducing $\overline{\text{S-T-CON}} \in$ co-NL-complete to it. Let (G, s, t) be a $\overline{\text{S-T-CON}}$ instance where $G = (V, E); s, t \in V; s \neq t$. We regard V as variables and output the following formula:

$$\varphi(G, s, t) = (s \vee s) \wedge (\neg t \vee \neg t) \wedge \bigwedge_{(u,v)\in E} (\neg u \vee v)$$

Let $S \subseteq V$ be the set of vertices reachable from s in G and $T \subseteq V$ be the set of vertices reachable from t in G^T.

Correctness:

($\Rightarrow$) Consider $\rho : V \mapsto \{true, false\}$, $\rho(v) = true$ for $v \in S$ and $\rho(v) = false$ for $v \notin S$. If t is not reachable from s, then ρ satisfies $\varphi(G, s, t)$ since $s \in S$, $t \in T \subseteq V - S$ and no edge $(u, v) \in E$ has $u \in S \wedge v \in V - S$.

($\Leftarrow$) Let ρ be a satisfying truth assignment for $\varphi(G, s, t)$; consider a potential path P in G: $s = v_1 \to v_2 \to \cdots \to v_k = t$. We have $\rho(s) = true$, $\rho(t) = false$ so there exists some $1 \leq i < k$ such that $\rho(v_i) = true$ and $\rho(v_{i+1}) = false$. But $(\neg\rho(u) \vee \rho(v))$ for all $(u, v) \in E$, hence $(v_i, v_{i+1}) \notin E$ and P is not a path from s to t in G.

Space Complexity:
Logarithmic, as we pass once over the input and have to remember only indices $\leq n$.

2. See Papadimitriou's book.

3. (a) We use dynamic programming:
Let $m(k, t)$ be the result of SUBSET-SUM$((a_1, \ldots, a_k), t)$ for $0 \leq k \leq n, 0 \leq t \leq s$.
Begin with $m(0, 0) = true$, $m(0, t) = false$ for $t > 0$;
Calculate the next column $m(k + 1, t) = m(k, t) \vee m(k, t - a_k)$[1].
The output is $m(n, S)$.
Time Complexity: We fill a $(n+1) \times (S+1)$ matrix, spending $O(\log S)$ time per cell[2], so the whole algorithm takes $\Theta(nS \log S)$ time.
Note: space complexity can be reduced from $O(nS \log S)$ to $O(S \log S)$ by keeping only the previous and current columns.

   (b) Recall that the size of the input (A, S) for SUBSET-SUM is $\log(S) + \sum_{a\in A} \log(a) = O(n \log a_{max} + \log S)$. The algorithm of 3a uses $\Theta(nS \log S) = \Theta(n2^{\log S} \log S)$ which is exponential regarding to the size of the input. Moreover, in the reduction that proves SUBSET-SUM NP-hardness we actually used $S = 2^{\Omega(n)}$.

[1]Define $m(k, r) = false$ for $r < 0$.
[2]For adding two numbers.

4. Clearly DS $\in NP$ since we can check whether $S \subset V$ dominates G in $O(|V|^4)$. We show that DS is NP-hard by reducing Vertex-Cover (VC) to it.

   Let $(G, k), G = (V, E)$ be an instance of VC. Let $I \subset V$ be the set of isolated vertices in G. Define $\varphi(G, k) = (G', k)$ where $G' = (V', E')$, $V' = V \cup E - I$, $E' = E \cup \{(v, e) | v \in e \in E\}$.

   Correctness:

   ($\Rightarrow$) Let $S \subseteq V, |S| = k$ be a vertex cover of G and let $S' = S - I$. For $v \in V - I$, there exists some edge $(u, v) \in E$ such that either $u \in S'$ or $v \in S'$; for $e = (u, v) \in E$, either $u \in S'$ or $v \in S'$. Hence, V' is dominated by S' of size $\leq k$.

   ($\Leftarrow$) Let $S \subseteq V', |S| \leq k$ be a dominating set of G' with minimal $|S \cap E|$. We claim that $S \subseteq V$: if some $e = (u, v) \in S \cap E$ exists, we could take either u or v to S instead of e, reducing $|S \cap E|$. The set remains a dominating set since $\{u, v, e\}$ is a triangle in G'. Now, for $e = (u, v) \in E \subseteq V'$ we have some $v \in S$ dominating e, so S is a vertex cover of size $\leq k$ in G.

   Space Complexity:
   Logarithmic, as we remember only indices $\leq n$.

5. Use a padding argument, analogous to the one used to scale up *Savitch's Theorem*.

$$L \in NSPACE(n) = NSPACE(\log 2^n) \Rightarrow L' \in NL \Rightarrow$$
$$L' \in \text{co-}NL \Rightarrow L \in \text{co-}NSPACE(\log 2^n) = \text{co-}NSPACE(n)$$

   This actually proves $NSPACE(f(n)) = \text{co-}NSPACE(f(n))$ for all proper complexity functions $f(n) \geq \log n$.

6. (a) We can solve 3-SAT in NL^*: given a formula φ and a witness y, we check that y encodes a valid truth assignment for φ and that each clause of φ is satisfied by y. We only need logarithmic space for indices, and obviously $\varphi \in$ 3-SAT if and only if we accept φ, y.

   Since any $L \in NP$ can be logarithmically reduced to 3-SAT, $NP \subseteq NL^*$.

   (b) We have $NL \subseteq P \subseteq NP \subseteq NL^*$. If $P \neq NP$, this containment is strict, so $NL \neq NL^*$.