Exercise #6

Proposed Solution
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1. We seek $c = \min_{A\subseteq V} c_A$ where $c_A = \frac{E(A,V-A)}{\min(|A|,|V-A|)}$; we have to show

   - (upper bound) $c \le \min_{A\subseteq V} c_A$ by showing some A such that $c_A = c$;
   - (lower bound) $c \ge \min_{A\subseteq V} c_A$ by showing that for all A, $c_A \ge c$.

   (a) $G = K_n$. For any $A \subseteq V$, $E(A, V-A) = |A||V-A|$, hence $c_A = \max(|A|, |V-A|)$. Therefore, $c = \min c_A = \lceil \frac{n}{2} \rceil$.

   (b) $G = \mathbb{Z}_2^n$, a 4-regular lattice. Consider $A = \{(i,j) | 0 \le i < n, 0 \le j < \frac{n}{2}\}$, $|A| = n\frac{n}{2}$ (of n^2 vertices). We have $E(A, V-A) = 4n$ since every vertex on the side boundary of the A has one neighbor in $V-A$, so $c \le c_A = \frac{4}{n}$ (for even n).

   For the lower bound, consider a set A of size $|A| \le \frac{n^2}{2}$. Assume without loss of generality that both A and $V-A$ are connected[1].

   By the pigeonhole principle, A has vertices with ≤ 2 neighbours in A (e.g., corner vertices)[2]. Any vertex of $V-A$ that has ≥ 3 neighbours in A can switch places with such a corner vertex, only decreasing $E(A, V-A)$, so we can arrange that both A is (almost) rectangular.

   Consider now a $x \times y$ rectangle. If $x, y < n$, it has $2(x+y)$ boundary vertices contributing $2(x+y)$ edges to $E(A, V-A)$, thus $c_A = \frac{2}{x} + \frac{2}{y} \ge \frac{4\sqrt{2}}{n}$; If $x = n$ or $y = n$ (both cannot happen as $xy = |A| \le \frac{n^2}{2}$) we have $E(A, V-A) = 2n$, so $c_A \ge \frac{2n}{|A|} \ge \frac{4}{n}$.

   In any case, $c_A \ge \frac{4}{n}$.

   (c) Consider $A = \{(0, x_2, \dots, x_n) | x_i \in \mathbb{Z}_2\}$, $|A| = 2^{n-1}$. We have $E(A, V-A) = 2^{n-1}$ since every vertex $(0, x_2, \dots, x_n) \in A$ has exactly one neighbor $(1, x_2, \dots, x_n) \in V-A$, so $c \le c_A = 1$.

   The proof of the lower bound is similar to 1b.

2. We assume $\Delta(G) = \Delta$ (independent on n). For a given vertex $w \in V$, let $U_i(w)$ be the set of vertices reachable from w in i steps. Obviously, $U_0(w) = \{w\}$, $U_{i+1} = U_i \cup N(U_i)$.

   Since G is a c-expander, $|E(U_i, V-U_i)| \ge c|U_i|$ (as long as $|U_i| \le \frac{n}{2}$). Every vertex in $N(U_i)$ has degree $\le \Delta$, so $|N(U_i)| \ge \frac{1}{\Delta}|E(U_i, V-U_i)| \ge \frac{c}{\Delta}|U_i|$ and $|U_{i+1}| = (1 + \frac{c}{\Delta})|U_i|$. Hence, for $i^* = 1 + \log_{1+\frac{c}{\Delta}} \frac{n}{2}$, $|U_{i^*}| > \frac{n}{2}$.

   Now, $U_{i^*}(u) \cap U_{i^*}(v) \neq \varnothing$ since both cover more than half of V, thus there exists a path connecting u and v of length $2i^* = O(\log n)$.

   Note that the above holds for directed graphs as well (consider the analoguous $V_i(w)$, the set of vertices from which w is reachable in i steps).

3. We reduce from Gap-4-NAE.

   Lemma. $\text{Gap}_{[a,b]}$-4-NAE $\le_P$ $\text{Gap}_{[a',b']}$-3-NAE for $a' = \frac{1+a}{2}, b' = \frac{1+b}{2}$.

[1] If A has more than one connected component, we can decrease $E(A, V-A)$ by moving a connected component of it near another connected component.

[2] Unless A is a rectangle of length or width n, but that case is covered as well in the analysis.

Proof. Given a 4-CNF formula φ with m clauses, we generate a 3-CNF formula φ' with $2m$ clauses by replacing each clause $C_i = l_1 \vee l_2 \vee l_3 \vee l_4$ by two clauses $C_i' = l_1 \vee l_2 \vee z_i$, $C_i'' = l_3 \vee l_4 \vee \neg z_i$ where z_i is a new variable. Obviously, the reduction is polynomial.

As seen in the 4-NAE $\leq_P$ 3-NAE reduction, a truth assignment NAE-satisfies C_i if and only if it can be extended to a truth assigment that NAE-satisfies both C_i' and C_i'' (otherwise exactly one of C_i', C_i'' is NAE-satisfied).

If $\varphi \in YES$, at least bm clauses can be NAE-satisfied in φ, so at least $b \cdot 2m + (1-b)m = b'2m$ clauses can be NAE-satisfied in φ', hence $\varphi' \in YES$.

If $\varphi \in NO$, at most am clauses can be NAE-satisfied in φ, so at most $a \cdot 2m + (1-a)m = a'2m$ clauses can be NAE-satisfied in φ', hence $\varphi' \in NO$. $\square$

Since $\textsc{Gap}_{[a,b]}$-4-NAE is known to be NP-hard for $a = \frac{7}{8} + \epsilon, b = 1$, we conclude that $\textsc{Gap}_{[\frac{15}{16}+\epsilon',1]}$-3-NAE is NP-hard as well, so unless $P = NP$ no polynomial algorithm can approximate 3-NAE to within c for any $c > \frac{15}{16}$.

4. (a) We reduce from $\textsc{Gap}$-E3-SAT.

   Lemma. $\textsc{Gap}_{[a,b]}$-E3-SAT $\leq_P$ $\textsc{Gap}_{[a',b']}$-E4-SAT for $a' = \frac{1+a}{2}, b' = \frac{1+b}{2}$.

   Proof. Given a 3-CNF formula φ with m clauses, we generate a 4-CNF formula φ' with $2m$ clauses by replacing each clause $C_i = l_1 \vee l_2 \vee l_3$ by two clauses $C_i' = l_1 \vee l_2 \vee l_3 \vee z$, $C_i'' = l_1 \vee l_2 \vee l_3 \vee \neg z$ where z is a new variable. Obviously, the reduction is polynomial. Without loss of generality, any truth assignment lets $z = true$, so $\varphi' = \varphi \wedge \bigwedge_{i=1}^{m} true$.
   If $\varphi \in YES$, at least bm clauses can be satisfied in φ, so at least $bm + m = b'2m$ clauses can be satisfied in φ', hence $\varphi' \in YES$.
   If $\varphi \in NO$, at most am clauses can be satisfied in φ, so at most $am + m = a'2m$ clauses can be satisfied in φ', hence $\varphi' \in NO$. $\square$

   Since $\textsc{Gap}_{[a,b]}$-E3-SAT is assumed to be NP-hard for $a = \frac{7}{8} + \epsilon, b = 1$, we conclude that $\textsc{Gap}_{[\frac{15}{16}+\epsilon',1]}$-E4-SAT is NP-hard as well, so unless $P = NP$ no polynomial algorithm can approximate E4-SAT to within c for any $c > \frac{15}{16}$.

   (b) For a random truth assignment, the probability of a single clause to be satisfied (†) is $\frac{15}{16}$, hence the expected number of satisfied clauses is $\frac{15}{16}m$ and there *exists* some truth assignment satisfying at least $\frac{15}{16}$ of the clauses. Therefore, a conditional expectation algorithm (similar to the one for E3-SAT) approximates E4-SAT to within $\frac{15}{16}$.

   (c) No, as (†) is no longer true. Example: only half of the clauses in $\varphi = (x \vee x \vee x \vee x) \wedge (\neg x \vee \neg x \vee \neg x \vee \neg x)$ may be satisfied.

5. (a) Consider any truth assignment ρ and its complement $\neg\rho$ (assigning $\neg\rho(x)$ to each x). Every 3-equation is satisfied by exactly one of $\rho, \neg\rho$, hence for every system of m 3-equations, at least $\frac{m}{2}$ can be satisfied by one of them. Therefore, the algorithm that tests ρ and returns either ρ or $\neg\rho$ is a polynomial 2-approximation.

   Another possible solution is by conditional expectation (as was shown in class for E3-SAT).

(b) As seen in 5a, $NO = \varnothing$ for this problem; therefore, the trivial *return-true* algorithm maps YES instances to *true* and NO instances (none exist) to $false$, as requested.

(c) Since YES instances can be identified by a polynomial-time algorithm (e.g., elimination over $\mathbb{Z}_2$), this algorithm maps YES instances to *true* and all other (i.e., don't care and NO instances) to $false$, as requested.

(d) We show that $\textsc{Gap}_{[\frac{1}{2}+\epsilon,1-\epsilon]}\text{-}\textsc{LinEq} \leq_P \textsc{Gap}_{[\frac{7}{8}+\epsilon',1-\epsilon']}\text{-3-SAT}$ for $\epsilon' = \frac{\epsilon}{4}$; the result follows.

Assume without loss of generality that all equations are of the form $l_1 \oplus l_2 \oplus l_3 = 1$ (by negating a literal and the free term, if needed).

Given a system E of m 3-equations, we generate a 3-CNF formula φ with $4m$ clauses by adding for each equation $e : l_1 \oplus l_2 \oplus l_3 = 1$ the four clauses generated by converting $(l_1 \leftrightarrow (l_2 \leftrightarrow l_3))$ to CNF:

$$C_e = (l_1 \vee l_2 \vee l_3) \wedge (l_1 \vee \neg l_2 \vee \neg l_3) \wedge (\neg l_1 \vee l_2 \vee \neg l_3) \wedge (\neg l_1 \vee \neg l_2 \vee l_3)$$

Obviously, the reduction is polynomial.

Observe that an assignment satisfies e if and only it satisfies all four 3-CNF clauses, and any assignment that doesn't satisfy e, satisfies exactly three of them.

If $E \in YES$, at least $(1-\epsilon)m$ equations in E can be satisfied, so at least $(1-\epsilon)4m+\epsilon\cdot 3m = (1-\epsilon')4m$ clauses can be satisfied in φ, hence $\varphi \in YES$.

If $E \in NO$, at most $(\frac{1}{2}+\epsilon)m$ equations in E can be satisfied, so at most $(\frac{1}{2}+\epsilon)4m + (\frac{1}{2}-\epsilon)3m = (\frac{7}{8}+\epsilon')4m$ clauses can be satisfied in φ, hence $\varphi \in NO$.