Solution for Exercise 7

1. The reduction is from $Gap_{[\frac{7}{8}+\varepsilon,1]}-3SAT$. We do it in two steps: first we reduce to $Gap_{[\alpha,1]}-t-occ-3SAT$ where t is some arbitrary (possibly large) constant. Then we reduce to $Gap_{[\alpha',1]}-3-occ-3SAT$.

   Both reductions are done by replacing multiple occurrences of a variable (say x) with a new set of variables (say $x_1,...,x_i$). We also enforce consistency by adding new clauses between those variables. The new clauses are of the form $(\overline{x_j}\vee x_k)$ which is equivalent to $(x_j\Rightarrow x_k)$. The scheme of which (ordered) couples form a new clause is given by a directed graph. In the first reduction this graph is an expander. In the second reduction this graph is a directed cycle. Note that in the first reduction we need expanders of various sizes, according to the number of occurrences of each variable.

   (a) 1st reduction: $Gap_{[\frac{7}{8}+\varepsilon,1]}-3SAT\le_L Gap_{[\alpha,1]}-t-occ-3SAT$.

   Recall that $G=(V,E)$ is c-expander if for every $A\subseteq V$, $|E(A,V\setminus A)|\ge c\cdot\min(|A|,|V\setminus A|)$. We call c the expansion factor of G. Also recall that there exists a polynomial-time algorithm that, given 1^i generates d-regular c-expander with i vertices (for some constants c and d).

   Given an instance Φ we construct Φ' as described above, using a d-regular c-expander for every variable, where the number of vertices of the expander is the number of occurrences of the variable.

   Note that if Φ contains m clauses then Φ' contains $m+3dm=(3d+1)m$ clauses as it contain all original clauses plus d clauses for every occurrence of every variable (due to the expander) and there is a total of $3m$ occurrences of variables.

   Note also that the number of occurrences of each variable in Φ' is exactly $t=d+1$.

   Correctness:

   ($\Rightarrow$) If Φ is satisfiable then so is Φ', as a satisfying assignment to Φ can be interpreted as a satisfying assignment to Φ' by assigning all variables $x_1,...,x_i$ the same value as x. Clearly all original clauses as well as new clauses are satisfied.

   ($\Leftarrow$) If Φ is at most $(\frac{7}{8}+\varepsilon)$ satisfiable then consider any assignment A to Φ'. Define an assignment A_{maj} to Φ to be the majority assignment; that is - A_{maj} assigns x the majority value of $x_1,...,x_i$. Clearly A_{maj} satisfies at most $(\frac{7}{8}+\varepsilon)m$ clauses of Φ (as does every assignment to Φ).

   We next count how many of the $(3d+1)m$ clauses of Φ' are not satisfied by A. Consider the $(\frac{1}{8}-\varepsilon)m$ unsatisfied clauses of Φ. For each of them, either the corresponding clause of Φ' is unsatisfied or it is satisfied, but then at least one of its variables does no correspond to the majority value. Let s be the number of times this happens.

   Consider now the new clauses of x. If there are s_x new variables (corresponding to x) that do not agree with $A_{maj}(x)$ then at least $c\cdot s_x$ of the clauses containing these variables are unsatisfied. This is due to the c-expander (consider the two sets of vertices in the expander: those who do correspond to the majority and those who don't. Due to the expansion properties of the expander many edges are from the 'don't' to the 'do').

   So the number of unsatisfied clauses is at least $(\frac{1}{8}-\varepsilon)m-s+c\cdot s$. As $c<1$ this is at least $c\cdot(\frac{1}{8}-\varepsilon)m$. Which means there are at most
   $(3d+1)m-c\cdot(\frac{1}{8}-\varepsilon)m$ satisfied clauses of the $(3d+1)m$ clauses of Φ' and so $\alpha=\frac{(3d+1)m-c\cdot(\frac{1}{8}-\varepsilon)m}{(3d+1)m}=\frac{(3d+1)-c\cdot(\frac{1}{8}-\varepsilon)}{3d+1}<1$

   (b) 2nd reduction: $Gap_{[\alpha,1]}-t-occ-3SAT\le_L Gap_{[\alpha',1]}-3-occ-3SAT$.

   Let Ψ be an input instance of size m and Ψ' the output instance of size m'. Every variable in Ψ occurs exactly t times (where t is some constant). Replacing x with $x_1,...,x_t$ and adding the clauses according to the directed cycle, we get an instance Φ' where each variable occurs exactly 3 times, and the number of clauses $m'=m+3\cdot m=4m$ (m original clauses and two additional clauses for each occurrence of a variable, of which there are $3m$, but each new clause was counted twice).

Correctness:
($\Rightarrow$) If Ψ is satisfiable then so is Ψ', as a satisfying assignment to Ψ can be interpreted as a satisfying assignment to Ψ' by assigning all variables $x_1, ..., x_t$ the same value as x. Clearly all original clauses as well as new clauses are satisfied.
($\Leftarrow$) If Ψ is at most α satisfiable then consider any assignment A to Ψ'.

Define an assignment A_{maj} to Ψ to be the majority assignment; that is - A_{maj} assigns x the majority value of $x_1, ..., x_t$. Clearly A_{maj} satisfies at most α clauses of Ψ (as does every assignment to Ψ).

We next count how many of the m' clauses of Ψ' are not satisfied by A. Consider the $1 - \alpha$ unsatisfied clauses of Ψ. For each of them, either the corresponding clause of Ψ' is unsatisfied or it is satisfied, but then at least one of its variables x_j does no correspond to the majority value. Let s be the number of times this happens. Note that is x_j has a "wrong" value then there is an unsatisfied clause of the new clauses of x (on the cycle of x). There are at most $\frac{t}{2}$ "wrong" values on each circle. Thus there are at least $\frac{(1-\alpha)m}{\frac{t}{2}}$ unsatisfied clauses.

Which means there are at most $m' - \frac{(1-\alpha)m}{\frac{t}{2}}$ satisfied clauses of the m' clauses of Ψ' and so $\alpha' = \frac{m' - (1-\alpha)m \cdot \frac{2}{t}}{m'} = 1 - (1-\alpha) \cdot \frac{2}{4 \cdot t} < 1$